Precalculus
Lesson 4.4: Properties of Rational Functions
Mrs. Snow, Instructor

When dealing with ratios of integers, they are identified as rational numbers. When we look at ratios of polynomials, we call them **rational functions.**

> A **rational function** is a function of the form
>
> $$\boxed{R(x) = \frac{p(x)}{q(x)}}$$
>
> where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Find the domain of the rational functions:

$$R(x) = \frac{2x^3 - 4}{x + 5}$$

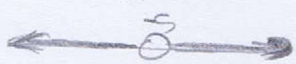
Denominator cannot = 0

$x + 5 \neq 0$

$x \neq -5$

D: $\{x \mid x \neq -5\}$

or



$(-\infty, -5) \cup (-5, \infty)$

$$R(x) = \frac{1}{x^2 - 4}$$

$(x+2)(x-2) \neq 0$

$x \neq 2 \quad x \neq -2$

D $\{x \mid x \neq -2, 2\}$

or

$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$$R(x) = \frac{x^3}{x^2 + 1}$$

all $\mathbb{R}$

D: $\{\mathbb{R}\}$

$$R(x) = \frac{x^2 - 1}{x - 1} \qquad \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

Yes, denominator factor out. Still

$R(x)$ is undefined if

$x = 1 \therefore$

D: $\{x \mid x \neq 1\}$

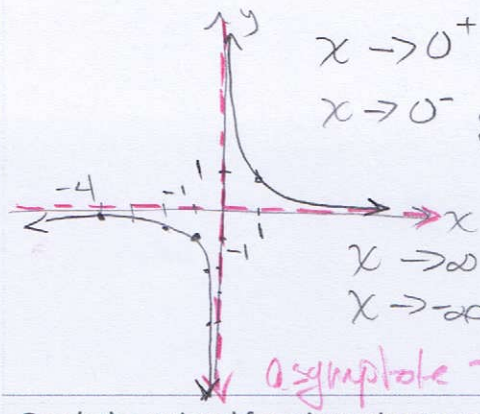
or $(-\infty, 1) \cup (1, \infty)$

Graph and analyze. What happens at $x = 0$? As x gets closer to 0? What happens as $x \to \infty$?

$$R(x) = \frac{1}{x}$$

x	y
-4	$-\frac{1}{4}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	$\frac{1}{-\frac{1}{2}} = -2$
$-\frac{1}{4}$	-4
0	UNDEF
$\frac{1}{4}$	$+4$

x	y
$\frac{1}{2}$	$+2$
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$



$x \to 0^+ \quad y \to \infty$

$x \to 0^- \quad y \to -\infty$

$x \to \infty \quad y \to 0$

$x \to -\infty \quad y \to 0$

asymptote $x = 0$

$$H(x) = \frac{1}{x^2}$$

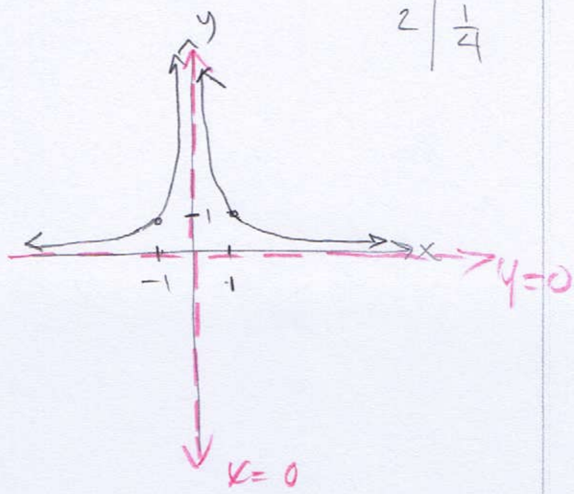
x	y
-2	$\frac{1}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{(\frac{1}{2})^2} = 4$
0	UNDEF
$\frac{1}{2}$	4
1	1
2	$\frac{1}{4}$

$x \to 0^+ \quad y \to \infty$

$x \to 0^- \quad y \to \infty$

$x \to \infty \quad y \to 0$

$x \to -\infty \quad y \to 0$



Asymptotes

Graph the rational function using transformations:

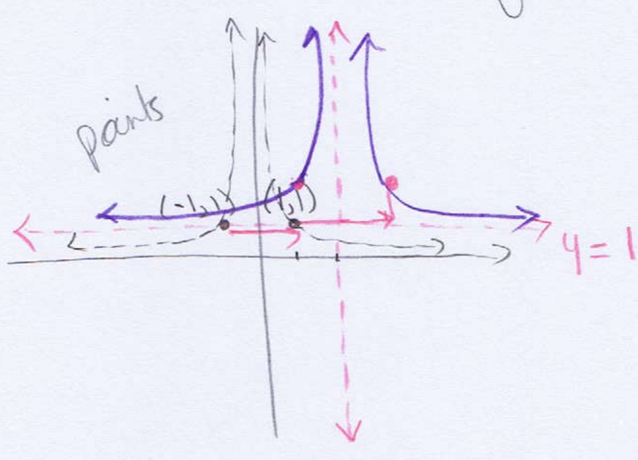
Parent $\frac{1}{x^2}$

$$R(x) = \frac{1}{(x-2)^2} + 1$$

$\longrightarrow 2 \quad \uparrow 1$

move asymptotes & pts $(-1, 1)$

$(1, 1)$

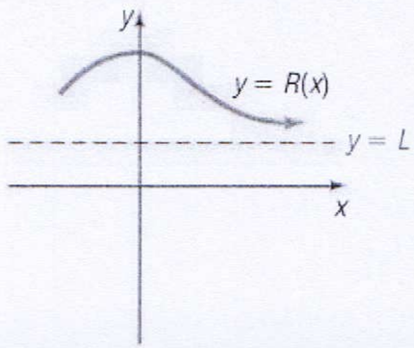


$\xrightarrow[2]{} \quad x = 2$

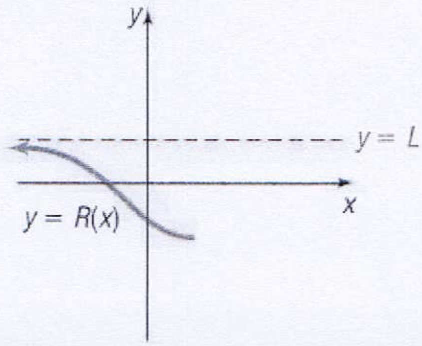
Asymptotes (pg. 219)

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.

Horizontal asymptotes:

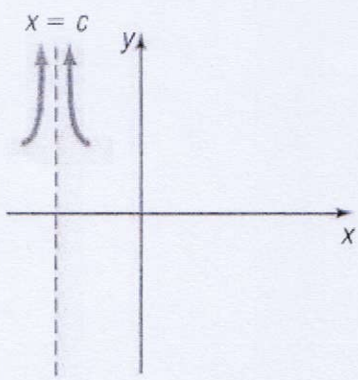


(a) End behavior:
As $x \to \infty$, the values of $R(x)$ approach L
[symbolized by $\lim_{x\to\infty} R(x) = L$].
That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

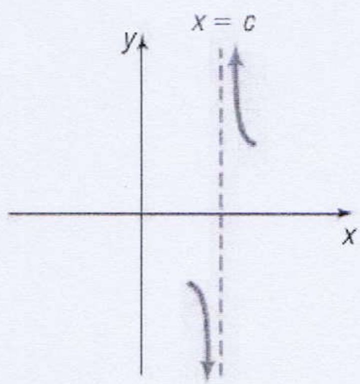


(b) End behavior:
As $x \to -\infty$, the values of $R(x)$ approach L
[symbolized by $\lim_{x\to-\infty} R(x) = L$].
That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

Vertical asymptotes:

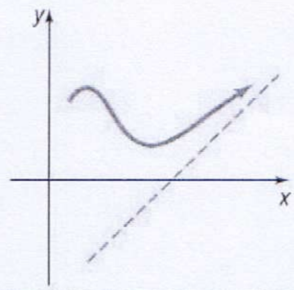


(c) As x approaches c, the values of $R(x) \to \infty$
[for $x < c$, this is symbolized by $\lim_{x\to c^-} R(x) = \infty$;
for $x > c$, this is symbolized by $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.



(d) As x approaches c, the values of $|R(x)| \to \infty$
[for $x < c$, this is symbolized by $\lim_{x\to c^-} R(x) = -\infty$;
for $x > c$, this is symbolized by $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

NOTE: Horizontal asymptotes may be intersected by the graph of a function! <u>The graph of a function will never intersect a vertical asymptote.</u>

There is also another type of asymptote, **OBLIQUE ASYMPTOTE**.	

Vertical Asymptotes

- Factor denominator and set it equal to zero. The values where the denominator goes to zero will be the vertical asymptotes; these are the domain restrictions and will graphically be seen as vertical asymptote(s).

Fid the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x+3}{x-1}$ VA: $x-1=0$

$x = 1$

(b) $R(x) = \dfrac{x}{x^2-4}$ → VA: $(x+2)(x-2)=0$

$x=-2,\ x=2$

(c) $H(x) = \dfrac{x^2}{x^2+1}$ VA $x^2+1=0$ No vertical asymptotes

$x^2=-1$

$x=\pm i$ ← No real zeros

(d) $G(x) = \dfrac{x^2-9}{x^2+4x-21}$ VA: $x^2+4x-21$

$(x+7)(x-3)=0$

$x=-7 \quad x=3$

Horizontal and Oblique Asymptotes

Last year we learned the little mnemonic of BOBO BOTN EATSDC... This is great, except there was one thing we did not discuss in Alg II and that was oblique asymptotes. So a new set of rules comes into play.

If the ratio for the rational function is **proper** (fraction); the degree of the numerator is less than the degree of the denominator, then it is a BOBO situation and the horizontal asymptote is $y = 0$.

Find the horizontal if one exists

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

$$\frac{\text{degree}=1}{\text{degree}=2}$$ proper

$\therefore$ HA: $\underline{\underline{y=0}}$

$$R(x) = \frac{p(x)}{q(x)} = \frac{\overbrace{a_n x^n}^{\text{degree } n} + \cdots a_1 x + a_0}{\underbrace{b_m x^m}_{\text{degree } m} + \cdots b_1 x + b_0} \begin{matrix} \leftarrow LC = a \\ \leftarrow LC = b \end{matrix}$$

Numerator has a degree of ***n*** and denominator has a degree of ***m***

If the ratio for the rational function is **improper;** the degree of the numerator is greater than or equal to the degree of the denominator: $n \geq m$. **Divide using long division:**
This will yield the following:

$$R(x) = \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

1. Degrees of numerator and denominator are equal $n = m$:
   This is what we described last year as EATSDC; the horizontal is described by a line equal to the ratio of the leading coefficients. $y = \frac{a}{b}$
2. Degree of numerator is one more than the denominator $n = m + 1$: Numerator greater by 1
   Divide, the quotient obtained is of the form of $ax + b$, the line $y = ax + b$ is the oblique asymptote.
3. Degree of the numerator is two more than the denominator $n = m + 2$: Numerator greater by 2
   Divide, the quotient is a polynomial for degree 2 or higher. R will have neither horizontal nor obliqui asymptotes. (The graph for large values of $|x|$ will behave like the graph of the quotient.)

no asymptotes.

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

$$\begin{array}{r} 3x + 3 \\ x^3 - x^2 + 1 \,\big)\overline{\,3x^4 + 0x^3 - x^2 + 0x + 0} \\ -\,(3x^4 - 3x^3 \quad\quad + 3x) \\ \hline 3x^3 - x^2 - 3x + 0x \\ -\,(3x^3 - 3x^2 \quad\quad + 3) \\ \hline 2x^2 - 3x - 3 \end{array}$$

Numerator degree greater by 1

$\Downarrow$

asymptote $y = ax + b$

$$H(x) = (3x+3) + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

oblique asymptote $\Rightarrow$ $y = 3x + 3$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

degrees equal

divide leading coefficients

$$y = \frac{8}{4}$$

$$y = 2$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

$$\begin{array}{r} 2x^2 - 1 \\ x^3-1 \overline{\big)\, 2x^5 + 0x^4 - x^3 + 0x^2 + 0x + 2} \\ -\cancel{2x^5} \qquad\qquad + 2x^2 \qquad\quad \\ \hline -\cancel{x^3} + 2x^2 + 2 \\ +\cancel{x^3} \qquad\quad \mp 1 \\ \hline 2x^2 + 1 \end{array}$$

denominator
numerator greater
by 2

$$G(x) = 2x^2 - 1 + \frac{2x^2+1}{x^3-1}$$

G(x) will
behave like
$y = 2x^2 - 1$

Ans no oblique no asymptotes

Precalculus

Lesson4.5: The Graph of a Rational Function

Mrs. Snow, Instructor

Calculators of course make graphing rational function much easier and quicker. However, we need to be proficient in using algebraic analysis to draw conclusions of the graph.

Analyze the rational function:

$$R(x) = \frac{x-1}{x^2-4}$$

1. Factor the numerator and denominator
   Find Domain :)
2. Write R in lowest terms, factor the numerator and denominator Deja vu!
3. Locate the intercepts of the graph:
   a. x-intercepts: determine the real zeros of the numerator
   b. y=intercepts: solve for $R(0)$
4. Locate the vertical asymptotes: find the zeros of the denominator
5. Locate the horizontal/oblique asymptotes: determine whether the function is proper or improper and follow the processes presented in lesson 4.4. Determine the points, if any, at which the graph of R intersects this asymptote. Graph the asymptotes using a dashed line. Plot any points at which the graph of R intersects the asymptote.
6. Graph R using a graphing calculator. And use the results from the analyses to graph R by hand.

(1) $R(x) = \dfrac{x-1}{(x+2)(x-2)}$

$D: \{x \mid x \neq -2, 2\}$

$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(3) x-intercept $(x, 0)$
when $y = 0$

$(x+2)(x-2)\,0 = \dfrac{x-1}{(x+2)(x-2)}(x+2)(x-2)$

$0 = x-1 \quad x = 1 \quad (1, 0)$

y incpt $(0, y)$

$y = \dfrac{0-1}{0-4} = \dfrac{1}{4} \quad \left(0, \dfrac{1}{4}\right)$

(4) VA: (denominator restriction)
$x = -2 \quad x = +2$

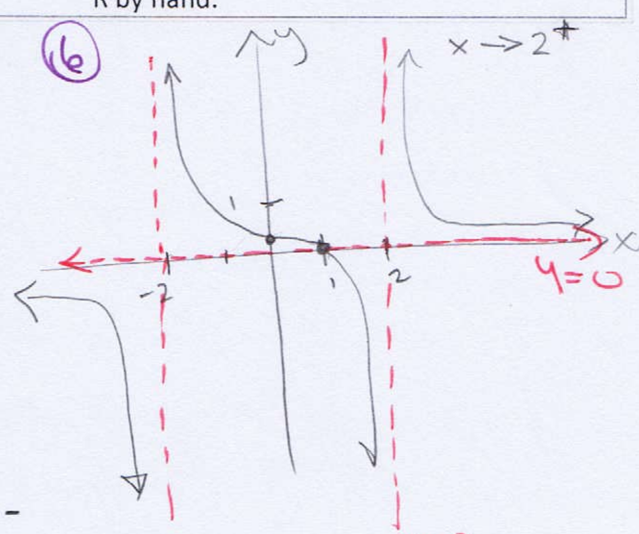
(5) HA/OA: degree
Bigger on Bottom
$y = 0$

Does it intersect the HA?

$R(x) = 0 \rightarrow 0 = \dfrac{x-1}{x^2-4}$

yes at $x = 1$

(6)



$\dfrac{x-1}{x^2-4} = \dfrac{+}{+} = +$

$x \to -2^-$

$\dfrac{x-1}{x^2-4} = \dfrac{-}{+} = -$

Analyze the rational function:

$$f(x) = \frac{x^2 - 1}{x} \qquad \frac{(x+1)(x-1)}{x}$$

1. Domain $(-\infty, 0) \cup (0, \infty)$

x-intcp $(x, 0)$ $\quad 0 = \frac{(x+1)(x-1)}{\cancel{x}}$

3. $(1, 0)$ $(-1, 0)$

y-intcp $(0, y)$ — undefined
none

4. VA $\quad x = 0$

5. degree numerator 1 greater

$$\begin{array}{r} x \\ x \,\overline{)\, x^2 + 0x - 1} \\ -\cancel{x^2} \\ \hline -1 \end{array}$$

$f(x) = x - \frac{1}{x}$

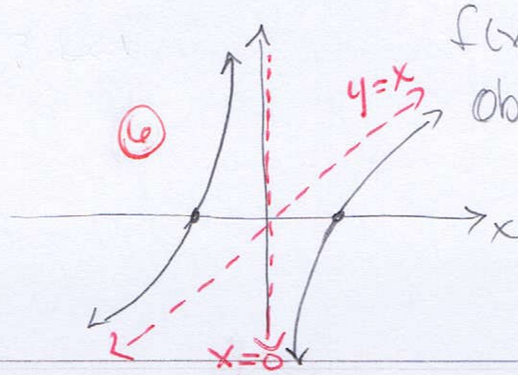
Oblique Asymptote
$y = x$

Intersect ?
$R(x) = x$

$\frac{x^2 - 1}{x} = x$

$\cancel{x^2 - 1 = x^2}$ NO

6.



Analyze the rational function:

$$f(x) = \frac{x^4 + 1}{x^2}$$

1. Domain
$(-\infty, 0) \cup (0, \infty)$

x-incpt $(x, 0)$

$0 = \frac{x^4 + 1}{x^2}$

$-1 = x^4$? no
complex x-intercept

3. $(0, y)$ $\quad \frac{0+1}{0}$ undef
no y-intercept

4. VA $\quad x = 0$

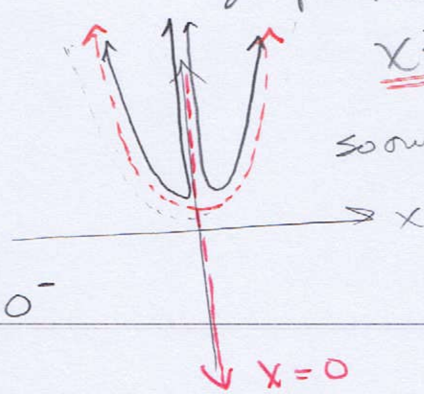
5. degree numerator 2 greater

$$\begin{array}{r} x^2 \\ x^2 \,\overline{)\, x^4 \quad + 1} \\ \cancel{x^4} \\ \hline +1 \end{array}$$

$f(x) = x^2 + \frac{1}{x^2}$

graph behaves like
x^2 for $|x| \to \infty$

so our graph



$x \to 0^-$

Analyze the rational function:

$$f(x)=\frac{3x^2-3x}{x^2+x-12}$$

EatsDC ⇒ HA $y=\frac{3}{1}=3$ (so no oblique)

① ② $\frac{3x(x-1)}{(x+4)(x-3)}=f(x)$

Domain

$x \neq -4,\ x \neq 3$

$(-\infty,-4)\cup(3,\infty)$

③ x-int $(x,0)$ $(0,0)$ $(1,0)$ & zeros

$0=\frac{3x(x-1)}{\cancel{(x+4)(x-3)}}$

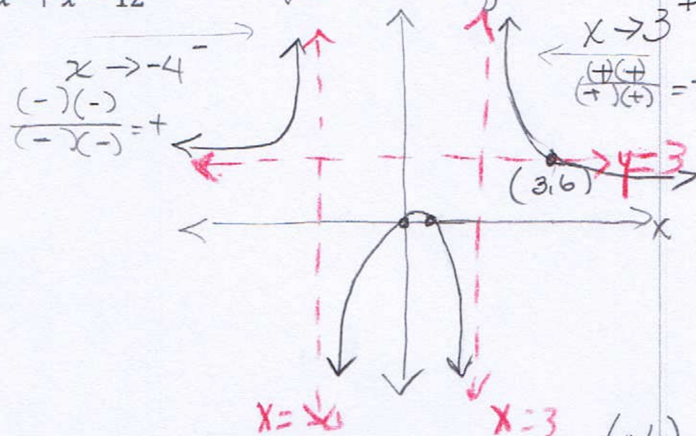
$3x=0 \quad x-1=0$

$x=0 \quad x=1$

y-int $(0,y)$ $(0,0)$

$y=0$

VA $x=-4,\ x=3$



does graph cross at $y=3$? $(3,6)$

so $f(x)=y \Rightarrow 3=\frac{3x^2-3x}{x^2+x-12}$

yes crosses at $x=6$

$3(x^2+x-12)=3x^2-3x$

$\cancel{3x^2}+3x-36=\cancel{3x^2}-3x$

$-36=-6x$

$6=x$

Analyze the rational function with a Hole

① ② Reduced

$$f(x)=\frac{2x^2-5x+2}{x^2-4}=\frac{(2x+1)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$ Hole $x=2$

domain:

-2 2

$(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

$f(x)=\frac{2x+1}{x+2}$ Hole: $(2,\frac{5}{4})$ $y=?$

$f(2)=\frac{2(2)+1}{2+2}=\frac{5}{4}$

③ x-int $(x,0)$ $(-\frac{1}{2},0)$ & zero

$0=\frac{2x+1}{\cancel{(x+2)}}$

$-\frac{1}{2}=x$

y-int $(0,y)$ $(0,\frac{1}{2})$

$y=\frac{0+1}{0+2}=\frac{1}{2}$

VA: $X=-2$

EATSDC ⇒ $y=\frac{2}{1}$ ∴ No Oblique

HA

so at $y=0$ does graph cross HA?

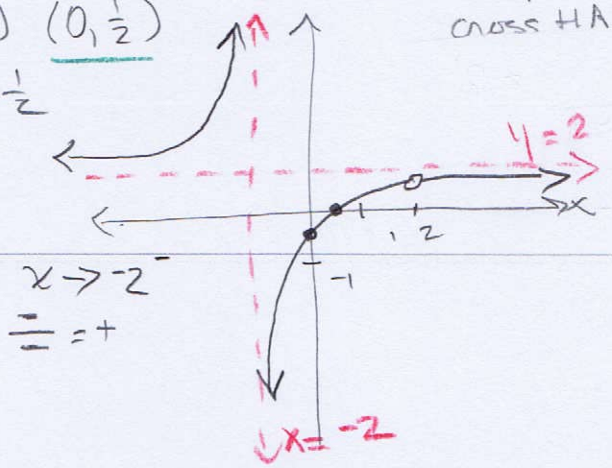
$2=\frac{2x-1}{x+2}$

$2(x+2)=2x-1$

$\cancel{2x}+4=\cancel{2x}-1$

$4=-1$ False

No

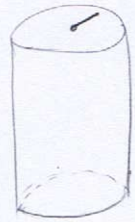


Finding the Least Cost

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters (cm^3), or $\frac{1}{2}$ liter. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢/per square centimeter (cm^2). The sides of the can are made of material that costs 0.02¢/cm^2.

volume

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?

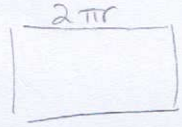


top & bottom: $A_T = (\pi r^2)2$ top & bottom

$C = .05(2\pi r^2) + .02(2\pi r h)$
?

2πr

h



$A = 2\pi r h$

Volume $= 500 = \pi r^2 h$

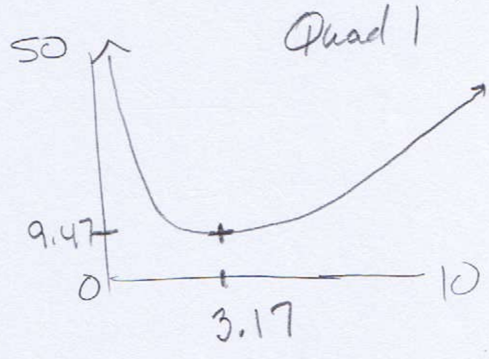
$\frac{500}{\pi r^2} = h$

$C = .10\pi r^2 + .04\cancel{\pi}\cancel{r}\left(\frac{500}{\cancel{\pi} r^{\cancel{2}}}\right)$

$C = \frac{r}{r} .1\pi r^2 + \frac{20}{r}$

(a) $C = \frac{.1\pi r^3 + 20}{r}$
cost

(b) (graphing cost (+y)
radius (+x)
Quad I



(c) least cost minimum
least cost at $r = 3.17$ cm

(d) least cost is 9.47 cents