Algebra II

Lesson4.5: 2x2 Matrices, Determinants and Inverses

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Data transferred over the internet is encoded or encrypted so that someone attempting to illegally access the data will find something that is unintelligible. One way to encrypt messages and data uses matrices and their **inverses**. Also, inverse matrices are used to simply solve matrix equations. The first step to calculating an inverse is to find the **determinant**. Using the determinant, we will be able to calculate the inverse.

Determinant – abbreviated **det** and symbol | |

For a 2×2 matrix, its determinant is found by subtracting the products of its diagonals:

Given a matrix A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are real numbers $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$\text{det A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example: Compute the determinant

$$A = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$$

$$\det A = \begin{vmatrix} -3 & 4 \\ 2 & -5 \end{vmatrix}$$

$(-3)(-5) - (4)(2)$

$15 - 8$

$\det A = 7$

$$B = \begin{bmatrix} 8 & -7 \\ 2 & 3 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 8 & -7 \\ 2 & 3 \end{vmatrix}$$

$(8)(3) - (-7)(2)$

$24 - (-14)$

$24 + 14 = 38$

$\det B = 38$

Matrix Inverse, but first....

What does an inverse do????? It undoes something. The additive inverse of x *is* $-x$. The multiplicative inverse of x *is* $\frac{1}{x}$ *or* x^{-1}. The inverses take us back to our identities: additive identity is 0 and the multiplicative identity is 1.

A lot of words here, but what we need to recognize is that the product of a matrix and its inverse will equal "1." The matrix identity is called, the **multiplicative identity matrix;** it is equivalent to "1" in matrix terminology. So, a matrix multiplied by **I** is equal to the matrix.

The identity matrix of a 2x2 is:

$$\mathbf{I_2} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Note: the identity matrix is identified with a capital **I** and a subscript indicating the dimensions. <u>The matrix identity is square</u>. Regardless of the dimensions of the matrix, the identity matrix consists of a diagonal of ones and the corners are filled in with zeros

Example: Multiply A by the identity matrix

$$\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 4\cdot 0 & 3\cdot 0 + 4\cdot 1 \\ -2\cdot 1 + 2\cdot 0 & -2\cdot 0 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$$

Multiplying a matrix with its identity matrix results in the original matrix!

Inverses: A number times its inverse (A.K.A. reciprocal) is equal to 1. A matrix times its inverse is also equal to "1."

Rule:

When two matrices are multiplied, and the product is the identity matrix, we say the two matrices are **inverses.**

Now we can set about to find inverses and verify if a matrix is an inverse of another.

It is a process, a pattern to follow and not that bad.

Example: Is **B** the inverse of **A**?:

$A=\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$	and $B=\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$	If B is the inverse, then AB should equal the identity matrix, does it?

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \overset{4-3}{(2)(2)+3(-1)} & \overset{-6+6}{2(-3)+3(2)} \\ \underset{2-2}{1(2)+2(-1)} & \underset{-3+4}{1(-3)+2(2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yes

B is the inverse of A

The notation for the inverse of a matrix is the matrix letter identity and a – 1 superscript, that is: $\mathbf{A}^{-1}$. In other words:

$\boldsymbol{A\,A^{-1} = A^{-1}A = I}$ (Remember I, identity matrix, is the "1" for matrices)

Well, how do we find the inverse? We do scalar multiplication with the value of $\frac{\mathbf{1}}{\mathbf{det}\,A}$ times a mixed-up version of matrix A!! Look below to see what I mean, it is not that bad.....

For matrix **A=** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **the inverse is:** **A**$^{-1}$**=** $\frac{\mathbf{1}}{\mathbf{det}\,A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ change sign

switch d & a

Note: det A in the denominator. Determinants may **not** be equal to 0 as $\frac{\mathbf{1}}{\mathbf{det}\,A}$ would be undefined.

So, **a matrix with a determinant of 0 has no inverse** and is called a **singular matrix**.

Example: Find the inverse of the matrix,

1. verify that $\det A \neq 0$
2. set up inverse equation *(note: switch a and d, and make c and b opposite sign)*

$A = \begin{bmatrix} -2 & 2 \\ 5 & -4 \end{bmatrix}$ $\det A = \begin{vmatrix} -2 & 2 \\ 5 & -4 \end{vmatrix} = (-2)(-4) - 2(5)$ $= 8 - 10 = -2 = \det A$ $A^{-1} = \frac{1}{-2}\begin{bmatrix} -4 & -2 \\ -5 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5/2 & 1 \end{bmatrix}$	$A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$ $\det A = \begin{vmatrix} 3 & 2 \\ 9 & 6 \end{vmatrix} = 18 - 18 = 0$ $\det A = 0$ No Inverse
$A = \begin{bmatrix} 6 & 3 \\ 5 & 8 \end{bmatrix}$ $\det A = \begin{vmatrix} 6 & 3 \\ 5 & 8 \end{vmatrix} = 48 - 15 = 33$ $A^{-1} = \frac{1}{33}\begin{bmatrix} 8 & -3 \\ -5 & 6 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 8/33 & -1/11 \\ -5/33 & 2/11 \end{bmatrix}$	$B = \begin{bmatrix} 0.5 & 2.3 \\ 3 & 7.2 \end{bmatrix}$ $\det B = \begin{vmatrix} .5 & 2.3 \\ 3 & 7.2 \end{vmatrix} = 3.6 - 6.9$ $= -3.3$ $B^{-1} = \frac{1}{-3.3}\begin{bmatrix} 7.2 & -2.3 \\ -3 & .5 \end{bmatrix} =$ $= \begin{bmatrix} 2.18 & .7 \\ 0.91 & -.16 \end{bmatrix}$

Well, what good is an inverse matrix? Glad you asked! In this class we will soon see that matrices are used for solving systems of simultaneous linear equations. In the world beyond McNeil, matrices are used for describing the quantum mechanics of atomic structure, designing computer game graphics, data encryption, analyzing relationships, and even plotting complicated dance steps!

If we are given a matrix equation such as : To solve a matrix equation, $\boldsymbol{AX} = \boldsymbol{B}$ for X, you simply multiply each side by A^{-1}. This process is in fact quite similar to solving an equation like $5x = 20$. To solve for x we multiply each side by 1/5, *the inverse of 5.*

To solve for a matrix:

1. $A^{-1}AX = A^{-1}B$, *(why?? A matrix times its inverse equals "1")*
   We are left with X on the left side which is what we are solving for. So: $\boldsymbol{X = A^{-1}B}$
2. **WARNING!!!** <u>You must always keep the order of the matrices uniform! $A^{-1}B$ is NOT the same as $B\,A^{-1}$!</u>

Example: Solve the matrix equation for X

$$\boldsymbol{X = A^{-1}B}$$

$$\begin{bmatrix}-2 & -5\\ 1 & 3\end{bmatrix} X = \begin{bmatrix}-2\\ 2\end{bmatrix}$$

A^{-1}: $\det A = -6-(-5) = -6+5 = -1$

$$\frac{1}{-1}\begin{bmatrix}3 & 5\\ -1 & -2\end{bmatrix} = \begin{bmatrix}-3 & -5\\ 1 & 2\end{bmatrix} = A^{-1}$$

$$\cancelto{I_2}{A^{-1}\begin{bmatrix}-2 & -5\\ 1 & 3\end{bmatrix}} X = \begin{bmatrix}-3 & -5\\ 1 & 2\end{bmatrix}\begin{bmatrix}-2\\ 2\end{bmatrix}$$

$$X = \begin{bmatrix}\overset{6}{-3(-2)} + \overset{-10}{(-5)2}\\ \underset{-2}{1(-2)} + \underset{+4}{2(2)}\end{bmatrix} = \begin{bmatrix}-4\\ 2\end{bmatrix}$$

1. Multiply each side by the inverse. Remember: inverse times matrix=inverse times matrix. **ORDER COUNTS!!!**

2. Multiplication by the inverse leaves X on the left side. Simplify the right side.

3. Calculate det A.

4. Using the determinant find A^{-1}

5. With A^{-1} simplify the right side of the equation to solve for X.

$$\begin{bmatrix}3 & -4\\ 4 & -5\end{bmatrix} X = \begin{bmatrix}0 & -22\\ 0 & -28\end{bmatrix}$$

A^{-1}: $\det A = -15-(-16) = 1$

$$A^{-1} = \frac{1}{1}\begin{bmatrix}-5 & 4\\ -4 & 3\end{bmatrix} = \begin{bmatrix}-5 & 4\\ -4 & 3\end{bmatrix}$$

$$\cancelto{I}{A^{-1}\begin{bmatrix}3 & -4\\ 4 & -5\end{bmatrix}} = \begin{bmatrix}-5 & 4\\ -4 & 3\end{bmatrix}\begin{bmatrix}0 & -22\\ 0 & -28\end{bmatrix} =$$

$$\begin{bmatrix}0+0 & 110-112\\ 0+0 & 88-84\end{bmatrix} = \begin{bmatrix}0 & -2\\ 0 & 4\end{bmatrix}$$

Solve the system of equations

The variables make up the variable matrix. The coefficients are the "A" matrix. The constants are the "B" matrix:

$$\begin{cases} 2x + 3y = 11 \\ 1x + 2y = 6 \end{cases}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

A^{-1}; $\det A = 4 - 3 = 1$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\underbrace{\cancel{A^{-1}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}}_{I}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 - 18 \\ -11 + 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{cases} 5a + 3b = 7 \\ 3a + 2b = 5 \end{cases}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

A^{-1}: $\det A = 10 - 3 = 1$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 14 - 15 \\ -21 + 25 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$