Calculus
Lesson 7.2: The Disk Method
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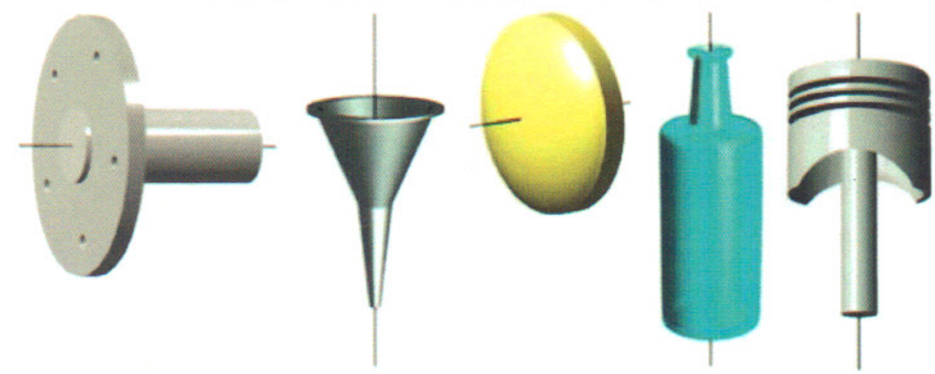
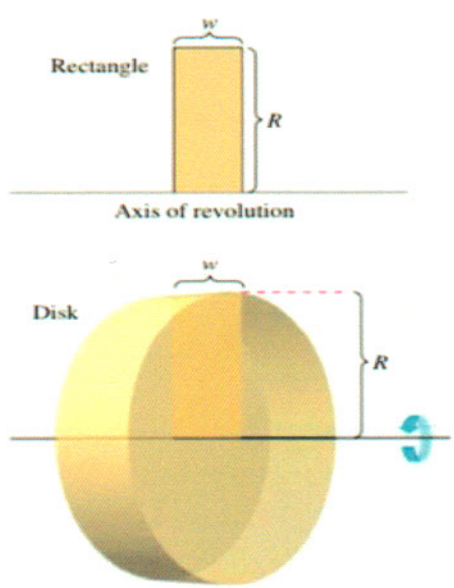
You have already learned that area is only one of the many applications of the definite integral. Another important application is its use in finding the volume of a three- dimensional solid. In this section you will study a particular type of three-dimensional solid—one whose cross sections are similar. Solids of revolution are used commonly in engineering and manufacturing. Some examples are axles, funnels, pills, bottles, and pistons.



Solids of revolution

The Disk Method
If a region in the plane is revolved about a line, the resulting solid is a solid of revolution, and the line is called the axis of revolution. The simplest such solid is a right circular cylinder or disk, which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle. ***WHAT?!?!?***



Volume of a disk: $\pi R^2 w$

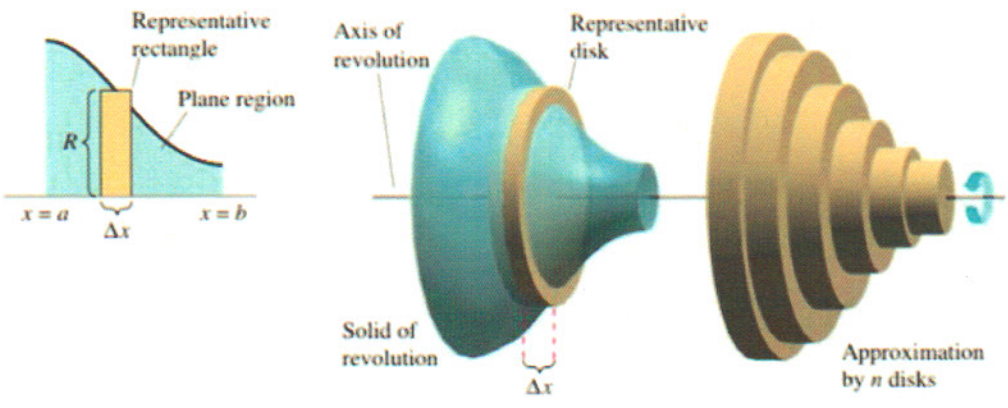
In the disk method, we take a solid object, slice it into many (infinite) small disks and add up the volumes of each disk using our geometry equation for a volume of a disk.

This goes back now to the Greek method of exhaustion, but here instead of adding rectangles, we take the rectangular cross section and rotate it around an axis to make a disk. What's the volume of a disk??

To determine the volume of a solid that has a cross section of a rectangle with a width of Δx and is rotated about an axis of revolution, here the axis, it will generate a volume of:

$$\Delta V = \pi R^2 \Delta x.$$

As the number of rectangles increases or as Δx gets smaller, the infinite number of rectangles generated take on the exact shape of the solid.



Disk method

The Disk Method

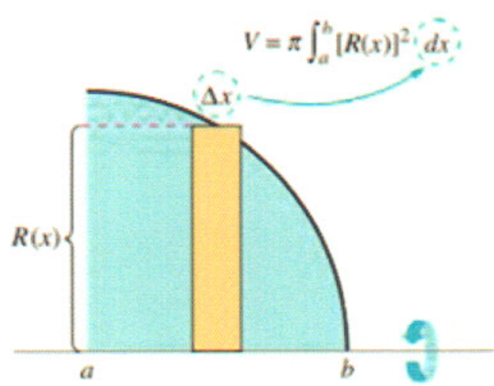
THE DISK METHOD

To find the volume of a solid of revolution with the **disk method,** use one of the following, as shown in Figure 7.15.

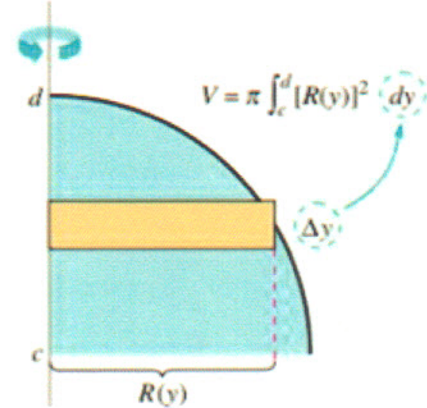
Horizontal Axis of Revolution	*Vertical Axis of Revolution*
Volume $= V = \pi \int_a^b [R(x)]^2\, dx$	Volume $= V = \pi \int_c^d [R(y)]^2\, dy$

What we need to notice is that the position of the rectangle with respect to the axis of revolution is perpendicular.

This is an important concept to understand as the method presented in lesson 7.3 uses a parallel relationship between the position of the rectangle and the axis of revolution.



Horizontal axis of revolution



Vertical axis of revolution

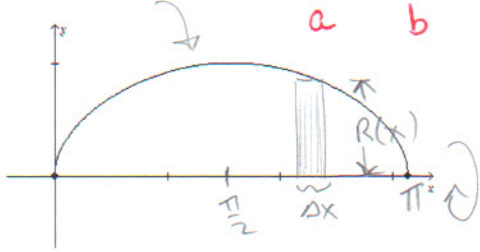
Using the Disk Method

- Find the volume of the solid formed by revolving the region bounded by the graph of f(x) and the x-axis about the x-axis.

$f(x) = \sqrt{\sin x}$ $\qquad 0 \le x \le \pi$

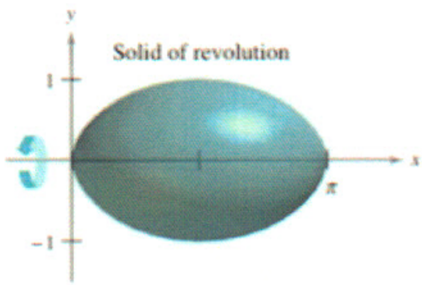
$\qquad\qquad\qquad\qquad a \qquad b$

draw the curve and identify R(x) & Δx



$$V = \pi \int_a^b (R(x))^2\,dx$$

$$V = \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx$$



$$= \pi \int_0^{\pi} \sin x\,dx$$

$$(-\pi)\cos x \Big|_0^{\pi}$$

$$-\pi\left(\cos\pi - \cos 0\right)$$

$$-\pi\,(-1 - 1)$$

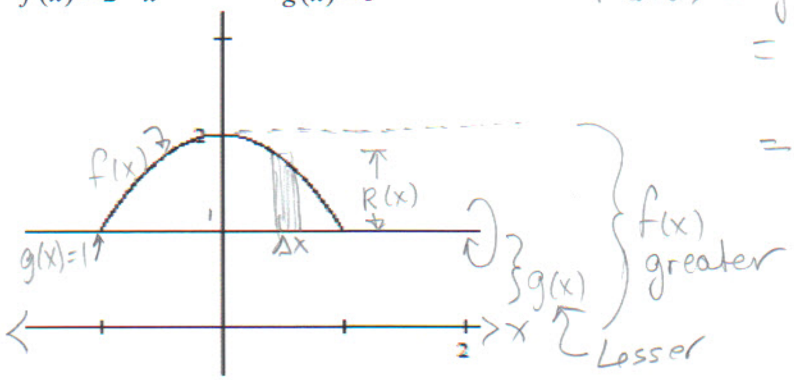
$$-\pi\,(-2) =$$

$$\boxed{2\pi = \text{volume}}$$

Revolving About a Line That is Not a Coordinate Axis

- Find the volume of the solid formed by revolving the region bounded by f(x) and g(x) and the line y=1.

$f(x) = 2 - x^2$ $\qquad$ $g(x) = 1$



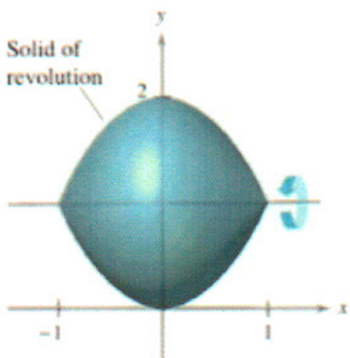
radius = greater − Lesser

$= f(x) - g(x)$

$= 2 - x^2 - 1 = R(x)$

$1 - x^2 = R(x)$

$$V = \pi \int_a^b (R(x))^2 \, dx$$



$$= \pi \int_{-1}^{1} (1-x^2)^2 \, dx$$

wait! we have symmetry so.....

Volume = volume

[1, 0] = [0, 1]

so we can double the area from 0 to 1 :)

$$V = 2\pi \int_0^1 (1-x^2)^2 \, dx$$

$(1-x^2)(1-x^2)$

$$= 2\pi \int_0^1 1 - 2x^{2^{+1}} + x^{4^{+1}} \, dx$$

$$= (2\pi)\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_0^1$$

$$= 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0)$$

$$= 2\pi\left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15}\right)$$

$$2\pi\left(\frac{8}{15}\right) =$$

$$\boxed{\frac{16\pi}{15} = \text{volume}}$$

The Washer Method

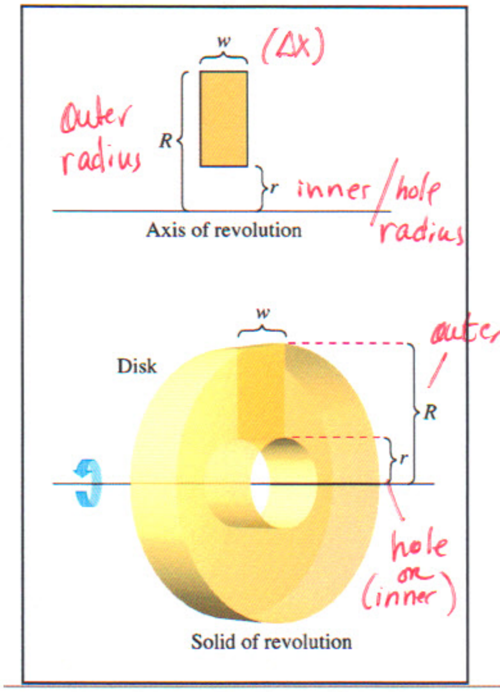
The disk method can be extended to cover solids of revolution with holes in the middle by replacing the representative disk with a representative washer. Revolve a rectangle about an axis only here we form a washer (a hole in the middle). We need to take into account the hole in the washer by subtracting the area of the hole from the area of the washer.

The Washer Method

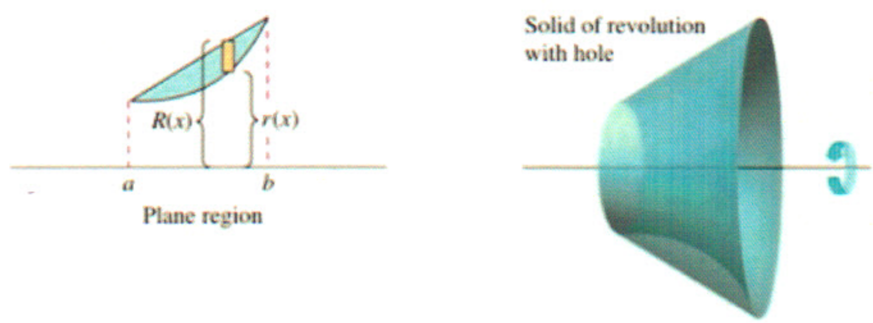
(outer)^2 – (hole)^2

$$V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx$$

$$V = \pi\int_a^b \left(\text{total radius}\right)^2 - \left(\text{hole radius}\right)^2 dx$$



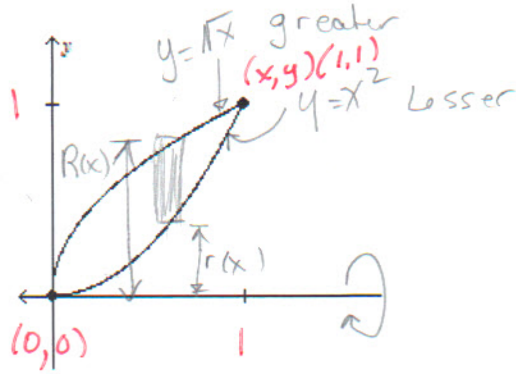
Note that the integral involving the inner radius represents the volume of the hole and is *subtracted* from the integral involving the outer radius.

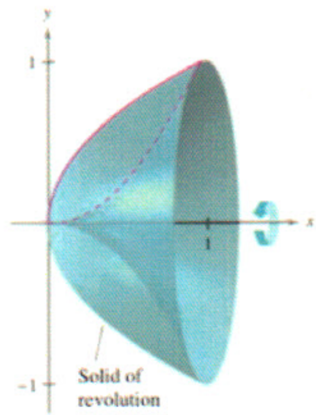


Using the Washer Method

- Find the volume of the solid formed by revolving the region bounded by the following graphs about the x-axis.

$y = \sqrt{x}$ $\qquad y = x^2$





Solid of revolution

$(x,y) = (0,0)$

$(x,y) = (1,1)$

$y = \sqrt{x} \quad y = x^2$

Set equal to each other

$\sqrt{x} = x^2$

$x = x^4$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0 \quad x^3 - 1 = 0$

$x = 1$

at $x = 1$ $y = 1$

at $x = 0$, $y = 0$

Total radius $= R(x) = \sqrt{x}$

hole radius $= r(x) = x^2$

$R^2 - r^2 =$ washer radius

$= (\sqrt{x})^2 - (x^2)^2 = x - x^4$

$$V = \pi \int_a^b (R(x))^2 - (r(x))^2 \, dx$$

$$V = \pi \int_0^1 x - x^4 \, dx$$

$$= (\pi) \left. \frac{x^2}{2} - \frac{x^5}{5} \right|_0^1$$

$$= (\pi) \left(\left(\frac{1}{2} - \frac{1}{5}\right) - 0 \right)$$

$$\pi \left(\frac{5}{10} - \frac{2}{10} \right) =$$

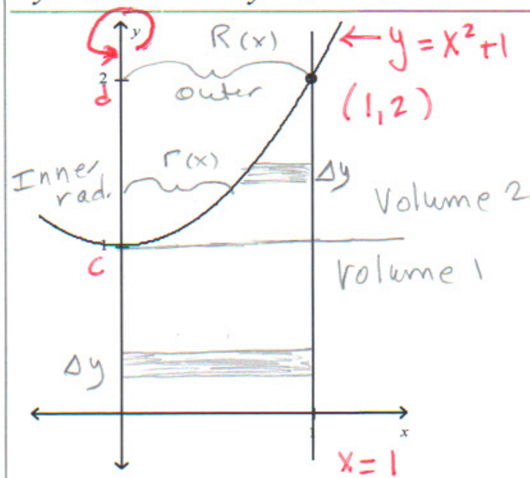
$$\boxed{\frac{3\pi}{10} = V}$$

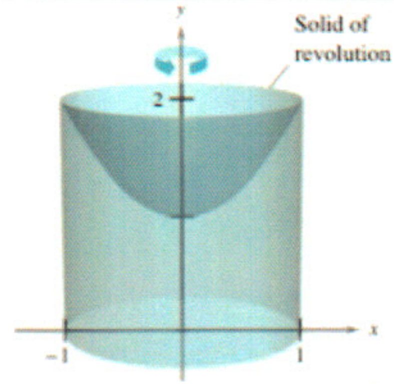
In each example so far, the axis of revolution has been horizontal and you have integrated with respect to x. In the next example, the axis of revolution is vertical and you integrate with respect to y. In this example, you need two separate integrals to compute the volume. y

Integrating with Respect to y, Two-Integral Case

- Find the volume of the solid formed by revolving the region bounded by the following graphs about the y-axis.

$y = x^2 + 1 \qquad y = 0 \qquad x = 1$





Total volume $= V_1 + V_2$

Remember

Outer radius − inner radius

$V = \pi \int_c^d (R(y))^2 - (r(y))^2 \, dy$

Volume 1 (no inner radius: r=0)

$= \pi \int_0^1 R^2 \, dy$

$= \pi \int_0^1 1^2 \, dy$

$= \pi \; y \Big|_0^1$

$(\pi)(1) = \boxed{\pi = V_1}$

Volume 2

$y = x^2 + 1$ solve for x

$\sqrt{y-1} = x$

$V = \pi \int_1^2 1^2 - (\sqrt{y-1})^2 \, dy$

$= \pi \int_1^2 1 - y + 1 \, dy$

$2 - y$

$= (\pi) \; 2y - \frac{y^2}{2} \Big|_1^2 =$

$\pi([4-2] - [2-\frac{1}{2}]) =$

$\pi(2 - \frac{3}{2}) = \pi(\frac{4}{2} - \frac{3}{2})$

$\boxed{V_2 = \frac{1}{2}\pi}$

$V_{TOTAL} = V_1 + V_2 = \pi + \frac{1}{2}\pi$

$V_T = \boxed{\frac{3}{2}\pi}$

Manufacturing

A manufacturer drills a hole through the center of a metal sphere of radius 5 inches, as shown in Figure 7.23(a). The hole has a radius of 3 inches. What is the volume of the resulting metal ring?

$$V = \pi \int_a^b (R(x))^2 - (r(x))^2 \, dx$$

$$= \pi \int_{-4}^{4} \left(\sqrt{25-x^2}\right)^2 - 3^2 \, dx$$

$$= 2\pi \int_0^4 25 - x^2 - 9 \, dx \quad ✩$$

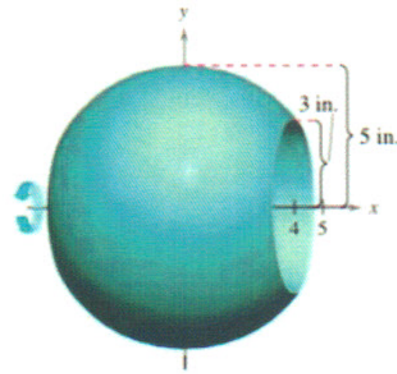
$$= 2\pi \int_0^4 16 - x^2 \, dx$$

$$= 2\pi \left(16x - \frac{x^3}{3}\right)\Big|_0^4$$

$$= (2\pi)\left(64 - \frac{64}{3} - 0\right)$$

$$(2\pi)\left(\frac{192}{3}\right) =$$

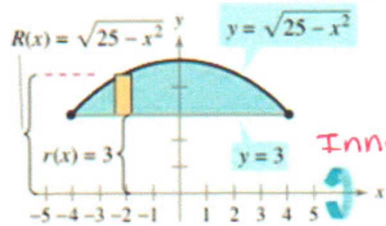
$$\boxed{\frac{256\pi}{3}} \text{ Volume}$$



Solid of revolution

(a)

Outer R(x)



Inner r(x)

Plane region

(b)

✩ Simplify the math

Symmetry about the y-axis so have bounds from 0 -> 4 & double the integral

Solids of Known Cross Sections

With the disk method, you can find the volume of a solid having a circular cross section whose area is $A = \pi R^2$. This method can be generalized to solids of any shape, as long as you know a formula for the area of an arbitrary cross section. Some common cross sections are squares, rectangles, triangles, semicircles, and trapezoids.

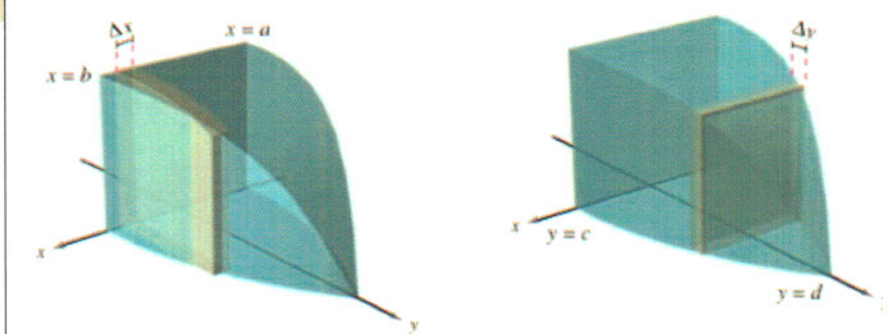
VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

$$\text{Volume} = \int_a^b A(x)\,dx.$$ See Figure 7.24(a).

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

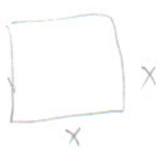
$$\text{Volume} = \int_c^d A(y)\,dy.$$ See Figure 7.24(b).



(a) Cross sections perpendicular to x-axis **(b)** Cross sections perpendicular to y-axis

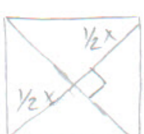
Let's have a quick review of some common areas:

1. A square with sides of length x



$A = x^2$

2. A square with diagonals of length x

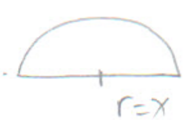


$s = \frac{\sqrt{2}}{2}x$

$A = s^2 = \left(\frac{\sqrt{2}}{2}x\right)^2$

$= \frac{1}{2}x^2 = A$

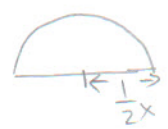
3. A semicircle of radius x



$A = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi x^2$

4. A semicircle of diameter x $\rightarrow r = \frac{1}{2}x$

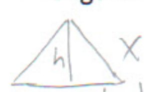


$A = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2$

$= \frac{1}{8}\pi x^2$

5. An equilateral triangle with sides of length x



$b = \frac{1}{2}x$

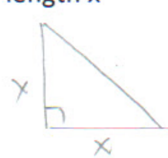
$h = \frac{\sqrt{3}}{2}x$

$A = \frac{1}{2}bh$

$= \left(\frac{1}{2}x\right)\left(\frac{\sqrt{3}}{2}x\right)$

$= \frac{\sqrt{3}}{4}x^2$

6. An isosceles right triangle with legs of length x



$A = \frac{1}{2}b(h)$

$= \frac{1}{2}x\,x$

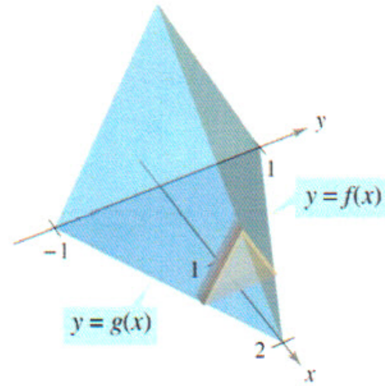
$= \frac{1}{2}x^2$

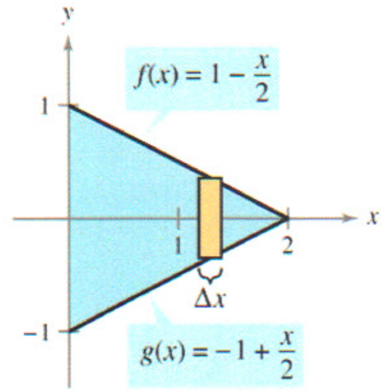
Triangular Cross Section

- Find the volume of the solid. The base of the solid is the regions bounded by the lines:

$f(x) = 1 - \frac{x}{2}$ $\qquad$ $g(x) = -1 + \frac{x}{2}$ $\qquad$ $x = 0$

- The cross sections perpendicular to the x-axis are equilateral triangles.





Equilateral triangle

$$A = \frac{\sqrt{3}}{4}(\text{base})^2$$

base = distance between 2 curves:

$$b = f(x) - g(x)$$
$$= 1 - \frac{x}{2} - \left(-1 + \frac{x}{2}\right)$$
$$= 1 - \frac{x}{2} + 1 - \frac{x}{2}$$
$$= 2 - x = \text{base}$$

$$A = \frac{\sqrt{3}}{4}(2-x)^2 =$$

$$A = \frac{\sqrt{3}}{4}(4 - 4x + x^2)$$

$$V = \int_a^b A(x)\,dx =$$

$$= \int_0^2 \frac{\sqrt{3}}{4}(4 - 4x + x^2)\,dx =$$

$$= \left(\frac{\sqrt{3}}{4}\right)\left(4x - 2x^2 + \frac{x^3}{3}\right)\Big|_0^2 =$$

$$\frac{\sqrt{3}}{4}\left(\cancel{8} - \cancel{8} + \frac{8}{3} - 0\right) = \frac{\sqrt{3}}{\cancel{4}}\left(\frac{\cancel{8}^{\,2}}{3}\right) = \boxed{\frac{2\sqrt{3}}{3} = V}$$