Calculus
Lesson 4.5: Integration by Substitution
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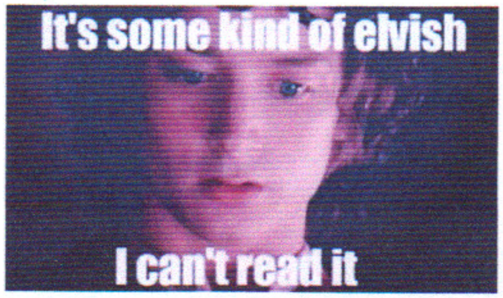
$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$



The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. In this section you will study techniques for integrating **composite functions.**

The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***"u*-substitution."**

- With pattern recognition you will perform the substitution mentally.
- With change of variables you will write the substitution steps.

THEOREM 4.13 ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Note that the composite function in the integrand has an *outside function f* and an *inside function g*.
Moreover, the derivative *g'*(*x*) is present as a factor of the integrand.

Outside function

$$\int f(g(x))\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

Inside function

Find these integrals.

→ Recognizing the $f(g(x))g'(x)$ pattern:

Find:

$$\int \underbrace{(x^2+1)^2}_{f(g(x))} \underbrace{(2x)}_{g'(x)} dx =$$

$$= \boxed{\frac{1}{3}(x^2+1)^3 + C}$$

Check

$$\frac{d}{dx}\frac{1}{3}(x^2+1)^3 + C =$$

$$(x^2+1)(2x) \checkmark$$

$$\int \underbrace{5}_{g'(x)} \underbrace{\cos 5x}_{f(g(x))}\, dx =$$

$$= \boxed{\sin 5x + C}$$

Check:

$$\frac{d}{dx} \sin 5x + C =$$

$$5\cos 5x \checkmark$$

You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\, dx = k\int f(x)\, dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple.

In such cases, you can multiply and divide by the necessary constant multiple

Find:

$$\int x\underbrace{(x^2+1)^2}_{f(g(x))}\, dx. = \qquad g'(x) = 2x$$

need a 2 so change look of integral NOT value: multiply by 1 $\times$ "1" or $\times \frac{2}{2}$

$$\int \frac{1}{2}(2x)(x^2+1)^2 dx =$$

$$\frac{1}{2}\int \underbrace{2x}_{g'(x)}\, \underbrace{(x^2+1)^2}_{f(g(x))} dx =$$

$$\frac{1}{2}\left(\frac{1}{3}\right)(x^2+1)^3 + C = \boxed{\frac{1}{6}(x^2+1)^3 + C}$$

With a formal change of variables, you completely rewrite the integral in terms of u and du (or any other convenient variable).

The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du = F(u) + C.$$

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Using change of variables find:

$\int \sqrt{2x-1}\,dx. =$

$u = 2x - 1$

$du = 2dx$

$\frac{1}{2}du = dx$

$\int \sqrt{u}\left(\frac{1}{2}\right)du =$

$\frac{1}{2}\int u^{1/2 \; +1}\,du =$

$\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)u^{3/2} + C$

$= \frac{1}{3}u^{3/2} + C$

$= \boxed{\frac{1}{3}(2x-1)^{3/2} + C}$

$\int x\sqrt{2x-1}\,dx =$ ✯

$u = 2x - 1$

$du = 2dx$

$\frac{1}{2}du = dx$

✯ also need x replaced:

solve for x:

$u = 2x - 1$

$u + 1 = 2x$

$\frac{u+1}{2} = x$

$\int \left(\frac{u+1}{2}\right)\left(u^{1/2}\right)\left(\frac{1}{2}\right)du =$

$= \frac{1}{4}\int (u+1)(u^{\frac{1}{2}})\,du =$

$= \frac{1}{4}\int u^{3/2 \; +1} + u^{1/2 \; +1}\,du =$

$= \frac{1}{\cancel{4}_2}\left[\frac{\cancel{2}}{5}u^{5/2} + \frac{\cancel{2}}{3}u^{3/2} + C\right]$

$= \boxed{\frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C}$

$\int \sin^2 3x \cos 3x \, dx =$

$u = \sin 3x$

$du = 3\cos 3x \, dx$

$\frac{1}{3} du = \cos 3x \, dx$

$\frac{1}{3} \int u^2 \, du =$

$\frac{1}{3}\left(\frac{1}{3}u^3\right) + c =$

$\frac{1}{9}u^3 + c =$

$\boxed{\frac{1}{9}\sin^3 3x + c}$

$\int 2\sec^2 x \,(\tan x + 3)\, dx =$

$u = \tan x + 3$

$du = \sec^2 x \, dx$

$\int 2u^{+1} \, du =$

$= \cancel{2}\left(\frac{1}{\cancel{2}}u^2\right) + c$

$= \boxed{(\tan x + 3)^2 + c}$

One of the most common *u*-substitutions involves quantities in the integrand that are raised to a power.

Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration:**

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Substitution and the General Power Rule

a. $\int 3(3x-1)^4\,dx = \int \underbrace{(3x-1)^4}_{u^4\ +1}\underbrace{(3)}_{du}\,dx = \frac{1}{5}u^5 + c = \boxed{\frac{1}{5}(3x-1)^5 + c}$

b. $\int (2x+1)(x^2+x)\,dx = \int \underbrace{(x^2+x)^1}_{u^1\ +1}\underbrace{(2x+1)}_{du}\,dx = \frac{1}{2}u^2 + c = \boxed{\frac{1}{2}(x^2+x)^2 + c}$

c. $\int 3x^2\sqrt{x^3-2}\,dx = \int \underbrace{(x^3-2)^{1/2}}_{u^{1/2}}\underbrace{(3x^2)}_{du}\,dx = \frac{2}{3}u^{3/2} + c = \boxed{\frac{2}{3}(x^3-2)^{3/2} + c}$

d. $\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \underbrace{-4x}_{du}\underbrace{(1-2x^2)^{-2}}_{u^{-2}}\,dx$ $= -1u^{-1} + c = -(1-2x^2)^{-1} + c$

$= \boxed{\frac{-1}{(1-2x^2)} + c}$

e. $\int \underbrace{\cos^2 x}_{u^2}\ \underbrace{\sin x\,dx}_{du}\ (-1) = -\frac{1}{3}u^3\,du$

$= \boxed{-\frac{1}{3}\cos^3 x + c}$

When using *u*-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable *u* rather than to convert the antiderivative back to the variable *x* and evaluate at the original limits.
This change of variables is stated explicitly in the following theorem.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Change of Variables

Evaluate $\int_0^1 x(\underbrace{x^2+1}_{u})^3\,dx$.

New Limits:

$u = x^2+1$ $\xrightarrow{\text{use}}$ $x=0$ $u = 0^2+1$, $u=0$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$ $x=1$ $u=1^2+1$, $u=2$

$\int_1^2 \frac{1}{2}u^3\,du =$

$= \frac{1}{2}\left(\frac{1}{4}\right)u^4\Big|_1^2$

$= \frac{1}{8}(2^4-1^4) = \frac{1}{8}(16-1)$

$= \frac{1}{8}(15)$

$= \boxed{\frac{15}{8}}$

OR (next page)

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$.

$u = 2x-1 \longrightarrow \frac{1}{2}(u+1) = x$

$du = 2\,dx$

$\frac{1}{2}du = dx$

Limits

$u = 2(1)-1 = 1$

$u = 2(5)-1 = 9$

$\frac{1}{4}\int_1^9 (u+1)u^{-1/2}\,du$

$\frac{1}{4}\int_1^9 u^{1/2}+u^{-1/2}\,du = \frac{1}{4}\left[\frac{2}{3}u^{3/2}+2u^{\frac{1}{2}}\right]_1^9$

$= \frac{1}{4}\left[\frac{2}{3}(27)+2(3)-\left[\frac{2}{3}+2\frac{3}{3}\right]\right] =$

$= \frac{1}{4}\left[18+6-\frac{8}{3}\right] = \frac{1}{4}\left[24-\frac{8}{3}\right]$

$= \frac{1}{4}\left[\frac{72}{3}-\frac{8}{3}\right] = \frac{1}{4}\left(\frac{64}{3}\right) = \boxed{\frac{16}{3}}$

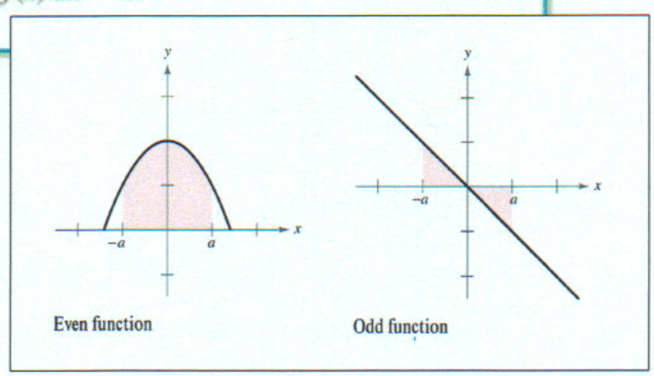
OR $\Rightarrow$

Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the *y*-axis or about the origin by recognizing the integrand to be an even or odd function

THEOREM 4.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.



Integration of an Odd Function

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^3 x\cos x + \sin x\cos x\right)dx = 0$$

Basic functions, nice to recognize even or odd.

Complicated functions – if we can graph on calculator, we can see symmetry

$$\int_0^1 x(x^2+1)^3\,dx$$

u-substitution and solve from previous page:

$$\int \frac{1}{2}u^3\,du = \frac{1}{2}\left(\frac{1}{4}\right)u^4$$

Change back to x

$$\frac{1}{8}(x^2+1)^4\Big|_0^1 =$$

$$\frac{1}{8}\left[2^4-1\right] =$$

$$\frac{1}{8}(16-1) = \boxed{\frac{15}{8}}$$

$$A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$$

$$u = \sqrt{2x-1}$$

trick

$$u^2 = 2x-1$$

derivative

$$2u\,du = 2\,dx$$

$$u\,du = dx$$

and solve for x:

$$u^2 = 2x-1$$

$$\frac{1}{2}(u^2+1) = x$$

new limits

$\boxed{x=1}$

$$u = \sqrt{2(1)-1}$$

$$u = 1$$

$\boxed{x=5}$

$$u = \sqrt{2(5)-1} = \sqrt{9}$$

$$u = 3$$

$$A = \int_1^3 \underbrace{\frac{1}{2}(u^2+1)}_{x}\,\underbrace{(\cancel{u}\,du)}_{dx}\,\underbrace{\cancel{u^{-1}}}_{\frac{1}{\sqrt{2x-1}}}$$

$$= \frac{1}{2}\left(\frac{1}{3}u^3+u\right)\Big|_1^3$$

$$= \frac{1}{2}\left(9+3-\frac{1}{3}-1\right)$$

$$= \frac{1}{2}\left(11-\frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{33}{3}-\frac{1}{3}\right)$$

$$= \boxed{\frac{16}{3}}$$

or revert back to x & solve

$$\frac{1}{2}\left(\frac{1}{3}(2x-1)^{3/2}+(2x-1)^{1/2}\right)\Big|_1^5 =$$

solve