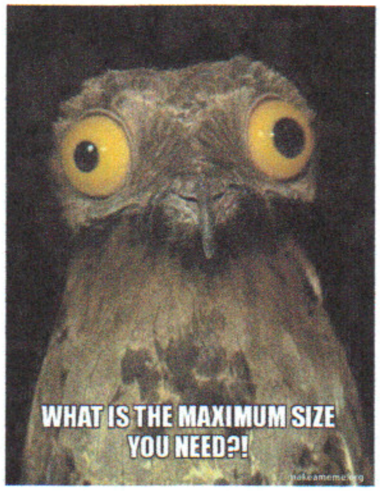
Calculus
Lesson 3.7 Optimization
Mrs. Snow, Instructor

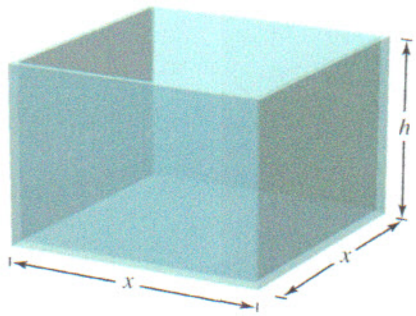


One of the most common applications of calculus involves the determining minimum or and maximum values. How often to you hear or read terms like: the greatest profit, least cost or time, the smallest size, the farthest distance, etc. Well that is what we are fixin' to do!

GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined.* If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



Open box with square base:
$S = x^2 + 4xh = 108$

What are we needing to maximize?

Volume – $V = x^2 h$

Also given: surface area

$$S = (area\ of\ base) + (area\ of\ four\ sides)$$
$$S = 108 = x^2 + 4xh$$

solving for h:

$$h = \frac{108 - x^2}{4x} = \frac{27}{x} - \frac{x}{4}$$

$$V = x^2\left(\frac{27}{x} - \frac{x}{4}\right)$$
$$V = 27x - \frac{x^3}{4}$$
$$\frac{dV}{dx} = 27 - \frac{3}{4}x^2 = 0$$
$$27 = \frac{3}{4}x^2$$
$$\frac{4}{\cancel{3}}(\overset{9}{\cancel{27}}) = x^2$$
$$36 = x^2$$
$$x = +6, \cancel{-6}$$

We are trying to maximize volume, so…

- → Write the volume as a function of one variable (x)
- → To maximize V, find the critical numbers of the volume function - take the derivative and set it equal to 0
- → determine the interval/doman over which there may be a solution for x

+ X must be non-negative
+ $V \geq 0$
+ as surface area is $108\,in^2$ & $A_{base} = x^2$ we know $x \leq \sqrt{108}$

$\therefore \quad 0 \leq x \leq \sqrt{108}$

Max volume

$V(0) = 0$

$V(6) = 108 \leftarrow$

$V(\sqrt{108}) = 0$

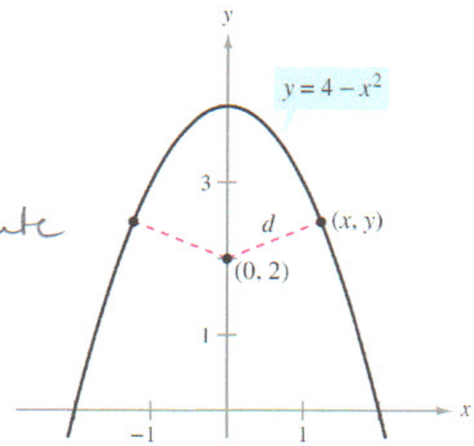
$\boxed{\text{dimensions} = 6 \times 6 \times 3 \text{ inches}}$

Finding the Minimum Distance

Which points on the graph *of* $y = 4 - x^2$ are closest to the point (0, 2) ?

→ Use the distance formula:
$d = \sqrt{(x-0)^2 + (y-2)^2}$ ← Substitute

→ because d is the smallest when the expression inside the radical is the smallest, you need to find the critical numbers of the radicand (f')



The quantity to be minimized is distance: $d = \sqrt{(x-0)^2 + (y-2)^2}$.

$d = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^2 + (2 - x^2)^2} =$

$(4 - 4x^2 + x^4)$

$= \sqrt{\underbrace{x^4 - 3x^2 + 4}_{f' \text{ expression under radical}}}$

$\frac{d}{dx} = 4x^3 - 6x = 0$

$2x(2x^2 - 3) = 0$

$\underline{x = 0}$ $\quad 2x^2 - 3 = 0$

$x^2 = \frac{3}{2}, \underline{x = \pm\sqrt{\frac{3}{2}}}$

	−2		−1		1		2
	$-\sqrt{\frac{3}{2}}$		0		$\sqrt{\frac{3}{2}}$		
(2x)	−		−		+		+
$(2x^2-3)$	+		−		−		+
	−		+		−		+
	↘	min	↗ max ↘		min	↗	

Minimum/closest at

$x = \pm\sqrt{\frac{3}{2}} \Rightarrow$

$$\boxed{\begin{array}{l}\text{minimum distance} \\ \left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)\left(+\sqrt{\frac{3}{2}}, \frac{5}{2}\right)\end{array}}$$

Finding Minimum Area

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

There are 2 things going on here:

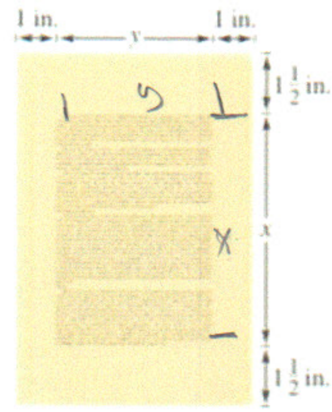
→ The area to be minimized:

$$A = (x+3)(y+2)$$

→ The printed area inside the margins:

$$(x)(y) = 24 \text{ in}^2 \qquad y = 24/x$$

- solve for y and substitute into the primary equation
- As area is positive, we need to calculate the critical numbers, differentiate with respect to x.
- domain $x > 0$



The quantity to be minimized is area: $A = (x + 3)(y + 2)$.

$$A = (x+3)\left(\frac{24}{x}+2\right) = 24 + 2x + \frac{72}{x} + 6$$

$$A = 30 + \frac{72}{x} + 2x$$

$$\frac{dA}{dx} = 2 - \frac{72}{x^2} = 0$$

$$2 = \frac{72}{x^2}$$

$$x^2 = \frac{72}{2}$$

$$x^2 = 36 \Rightarrow x = \cancel{-6}, 6$$

not in domain

$-$ 6 $+$

min

$$\boxed{\text{page dimensions: } 9 \times 6 \text{ inches}}$$

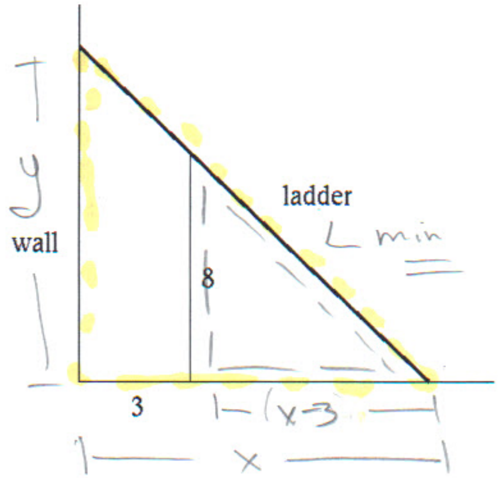
$$x + 3 = 6 + 3 = 9$$

$$y = \frac{24}{6} = 4$$

$$y + 2 = 6$$

The next couple pages show the derivative using the square root

Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall, which is 3 ft. behind the fence.



$L = \sqrt{x^2+y^2}$ minim.

Similar Δ.

$\frac{y}{x} = \frac{8}{3-x} \rightarrow \boxed{y = \frac{8x}{x-3}}$

$L = \sqrt{x^2 + \left(\frac{8x}{3-x}\right)^2} = \sqrt{x^2 + \frac{64x^2}{(x-3)^2}}$

again minimize the stuff under the radical

$f = x^2 + \frac{64x^2}{(x-3)^2}$

$\frac{d}{dx} = 2x + \frac{128x(x-3)^{\cancel{2}} - 64x^2(2)\cancel{(x-3)}}{(x-3)^{\cancel{4}\,3}}$

$= 2x + \frac{128x(x-3) - 128x^2}{(x-3)^3}$

$= 2x + \frac{128x(\cancel{x}-3-\cancel{x})}{(x-3)^3}$

$= 2x + \frac{128x(-3)}{(x-3)^3} = 2x - \frac{384x}{(x-3)^3} = 0$

$2\cancel{x} = \frac{384\cancel{x}}{(x-3)^3} \Rightarrow \cancel{2} = \frac{\cancel{384}\;192}{(x-3)^3}$

$(x-3)^{\cancel{3}\,\frac{1}{3}} = (192)^{\frac{1}{3}}$

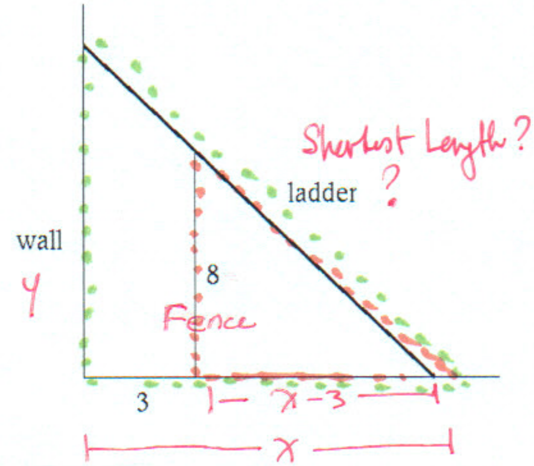
$x = (192)^{1/3} + 3 = 5.77 + 3 \qquad x = 8.77'$

$L = \sqrt{8.77^2 + 12.16^2} = \sqrt{244.77} = 14.99' \qquad y = \frac{8(8.77)}{8.77-3} = 12.16'$

$\boxed{15' \text{ Ladder}}$

With the square root → not very fun!

Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall, which is 3 ft. behind the fence.



$L = \sqrt{x^2+y^2} \Leftarrow$ minimize

y?? similar Δ

$$\frac{y}{x} = \frac{8}{x-3}$$

$$y = \frac{8x}{x-3} \quad \therefore$$

$$L = \sqrt{x^2 + \left(\frac{8x}{x-3}\right)^2} \quad \text{minimize so take derivative}$$

$$L = \left(x^2 + \frac{64x^2}{(x-3)^2}\right)^{1/2}$$

factor out $(x-3)(64x)$

$$L' = \frac{1}{2}\left(x^2+\frac{64x^2}{(x-3)^2}\right)^{-1/2}\left(2x + \frac{128x(x-3)^2 - 2(x-3)(64x^2)}{(x-3)^4}\right)$$

$$= \frac{1}{2}\left(x^2+\frac{64x^2}{(x-3)^2}\right)^{-1/2}\left(2x + \frac{64x\cancel{(x-3)}\left[2(x-3)-2x\right]}{(x-3)^{\cancel{4}\,3}}\right)$$

$$= \underbrace{\frac{1}{2}\left(x^2+\frac{64x^2}{(x-3)^2}\right)^{-1/2}}_{\text{move to denom}}\left(2x + \frac{64x\left[\cancel{2x}-6-\cancel{2x}\right]}{(x-3)^3}\right)$$

factor out $2x$ — Simplified → Set $=0$ to find minimum

$$= \frac{2x + \frac{64x(-6)}{(x-3)^3}}{2\left(x^2+\frac{64x^2}{(x-3)^2}\right)^{1/2}} = \frac{\cancel{2}x\left(1-\frac{192}{(x-3)^3}\right)}{\cancel{2}\left(x^2+\frac{64x^2}{(x-3)^2}\right)^{1/2}} = 0$$

Set numerator $= 0$

Solve for x → (next page cont)

(Minimum Ladder Length Example - $\boxed{\text{cont}}$)

$$\frac{x\left(1-\frac{192}{(x-3)^3}\right)}{\cancel{\left(x^2+\frac{64x^2}{(x-3)^3}\right)^{1/2}}}=0$$

$$x\left(1-\frac{192}{(x-3)^3}\right)=0$$

$\cancel{x=0}$ or $1-\dfrac{192}{(x-3)^3}=0$

Required $x>3$

$$1=\frac{192}{(x-3)^3}$$

$$(x-3)^3=192$$

$$x-3=\sqrt[3]{192}$$

$$x=\sqrt[3]{192}+3$$

$$x\approx 8.77\text{ft}$$

So what is the ladder length?

$L=\sqrt{x^2+y^2}$; $y=\dfrac{8x}{x-3}=\dfrac{8(8.77)}{8.77-3}=12.16\text{ft}$

$$L=\sqrt{8.77^2+12.16^2}$$

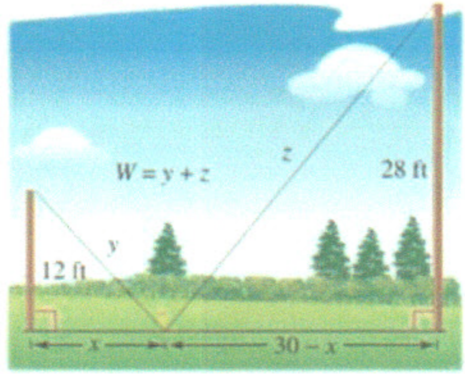
$$=\sqrt{224.77}$$

$\boxed{L=14.99\text{ft}}$ $\boxed{\text{so buy a 15 ft ladder}}$

An extra problem for you to study :)

Finding the Minimum Length
Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

- → Differentiate W with respect to x
- → calculate the critical values.
- → the interval for $x = [0,30]$



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.

$W = y + z$

$y^2 = 12^2 + x^2$

$y = \sqrt{144 + x^2}$

$z^2 = (30-x)^2 + 28^2 = 900 - 60x + x^2 + 784$

$z = \sqrt{x^2 - 60x + 1684}$

$W = (144 + x^2)^{1/2} + (x^2 - 60x + 1684)^{1/2}$ Chain Rule!

$$\frac{dW}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{144+x^2}} + \frac{\cancel{2}x - \cancel{60}\,30}{\cancel{2}\sqrt{x^2-60x+1684}}$$

$$= \frac{x}{\sqrt{144+x^2}} + \frac{x-30}{\sqrt{x^2-60x+1684}} = 0$$ & cross multiply

$\left(x(x^2-60x+1684)^{1/2}\right)^2 = \left((30-x)(144+x^2)^{1/2}\right)^2$

$x^2(x^2-60x+1684) = (30-x)^2(144+x^2)$

$\cancel{x^4} - \cancel{60x^3} + 1684x^2 = (900 - 60x + x^2)(144 + x^2)$

$= 129600 + 900x^2 - 8640x - \cancel{60x^3} + 144x^2 + \cancel{x^4}$

$640x^2 + 8640x - 129600 = 0$

$320(2x^2 + 27x - 405) = 0$

$320(2x^2 - 18x + 45x - 405) = 0$

$320(2x(x-9) + 45(x-9))$

$320(2x+45)(x-9) = 0$

$\cancel{x = \frac{-45}{2}}$ not neg. $x = 9$

$W(0) = 53.03$

$\boxed{W(9) = 50}$

$W(30) = 60.3$

minimum $x = 9$ ft

Stake wire 9' from 12' pole

An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the circle to enclose the maximum total area?

→ Total area is given by:

$$A = (area\ of\ square) + (area\ of\ circle)$$

$$A = x^2 + \pi r^2$$

→ Areas are formed by a 4 ft. length of wire, so we define the perimeter:

$$4 = (P\ of\ square) + (P\ of\ circle)$$

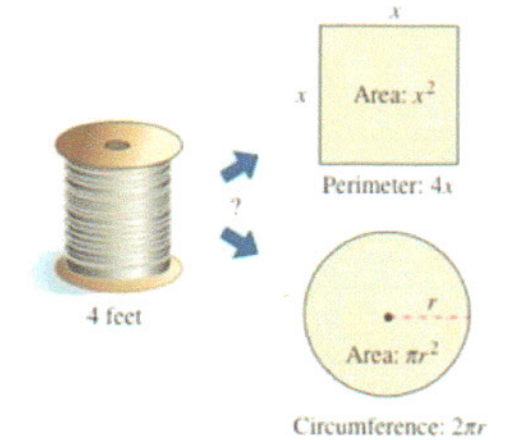
$$4 = 4x + 2\pi r$$

- Solve for r, substitute into the primary equation
- The domain of x is restricted by the fact that first, the length of the sides of the square must be greater than 0. Second, as we are limited to 4 ft. of wire and we have four sides of a square, so x can be no greater than 1.

$$D: 0 \leq x \leq 1$$

→ Differentiate A with respect to x

→ Find the critical number(s) in the domain



The quantity to be maximized is area: $A = x^2 + \pi r^2$.

$$P = 4 = 4x + 2\pi r$$

$$\frac{\cancel{4}^{2} - \cancel{4}^{2}x}{\cancel{2}\pi} = r = \frac{2-2x}{\pi}$$

$$A = x^2 + \pi\left(\frac{2-2x}{\pi}\right)^2 = x^2 + \cancel{\pi}\left(\frac{4-8x+4x^2}{\pi \cancel{\pi}}\right)$$

$$A = x^2 + \frac{4}{\pi} - \frac{8x}{\pi} + \frac{4}{\pi}x^2$$

$$\frac{dA}{dx} = 2x - \frac{8}{\pi} + \frac{8}{\pi}x = 0$$

$$\left(2 + \frac{8}{\pi}\right)x = \frac{8}{\pi}$$

$$x = \left(8/\pi\right) / \left(2 + \frac{8}{\pi}\right) \approx .56\ ft$$

$$[0, 1]$$

$$A(0) = 1.273$$

$$A(.56) = .56$$

$$A(1) = 1$$

Max area is at $x = 0$ → no square only the circle.