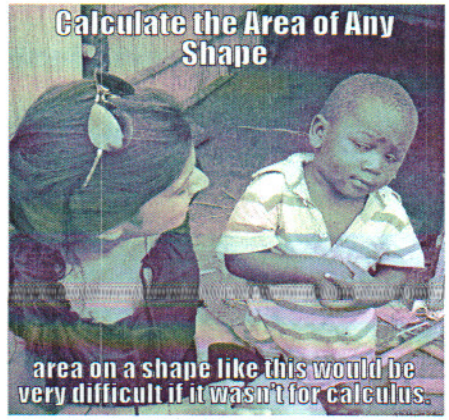
Calculus
Lesson 4.2: Area
Mrs. Snow, Instructor



In this section we will: review sigma notation and evaluate a sum, understand the concept of area, approximate the area of a plane region and find the area of a plane region using limits.

SIGMA NOTATION

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Examples of Sigma Notation

stop at $i = 6$

a. $\sum_{i=1}^{6} i = 1+2+3+4+5+6$

stop at $i = 5$

b. $\sum_{i=0}^{5} (i+1) = (0+1)+(1+1)+(2+1)+(3+1)+(4+1)+(5+1)$

c. $\sum_{j=3}^{7} j^2 = 3^2+4^2+5^2+6^2+7^2$

d. $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1)+\frac{1}{n}(2^2+1)+\frac{1}{n}(3^2+1)+\ldots$

$= \left(\frac{1}{n}\right)\left[(1^2+1)+(2^2+1)+(3^2+1)+\ldots\right]$

e. $\sum_{i=1}^{n} f(x_i)\,\Delta x = \Delta x\,(f(x_1)+f(x_2)+f(x_3)+\ldots)$

common factor

From precalculus we know:

$$1.\quad \sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$$ ← coeffient

$$2.\quad \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

THEOREM 4.2 SUMMATION FORMULAS

Know!!

* $\sum_{k=1}^{n} c = nc$

* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Evaluating a Sum

Evaluate

$\frac{1}{n^2}$ is a coefficient

$$\sum_{k=1}^{n} \frac{k+1}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k+1$$

separate summations & evaluate using above formulas

$$= \frac{1}{n^2}\left[\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + n(1)\right]$$

rearrange factors & distribute $\frac{1}{n^2}$, simplify

$$= \frac{1}{n^2}\left(\frac{n(n+1)}{2} + n\right)$$

$$= \left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{2n}\right) + \frac{\cancel{n}}{n^{\cancel{2}}}$$

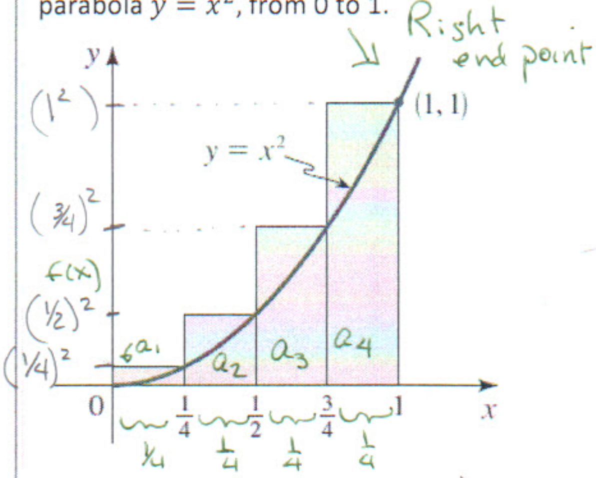
(common denom)

$$= \frac{n+1}{2n} + \frac{1}{n}\frac{2}{2}$$

$$= \frac{n+1+2}{2n} = \boxed{\frac{n+3}{2n}}$$

In geometry we found area of polygons. We had set formulas such as the area of a rectangle is length times width. A triangular area is found by calculating ½ the length of the base times the height, and so on. Calculus is used to deal with area problems that have regions containing curved boundaries. Here we can go back to our simple formula for the area of a rectangle and use it to estimate the area of a region under a curve.

Estimating an Area Using Rectangles: Use rectangles to estimate the area under the parabola $y = x^2$, from 0 to 1.



area = (width)(height)
= $(\Delta x)(f(x))$

$a_1 = \frac{1}{4}\left(\frac{1}{4}\right)^2 =$

$a_2 = \frac{1}{4}\left(\frac{1}{2}\right)^2$

$a_3 = \frac{1}{4}\left(\frac{3}{4}\right)^2$

$a_4 = \frac{1}{4}(1)^2$

$A_{Total} = \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right)$

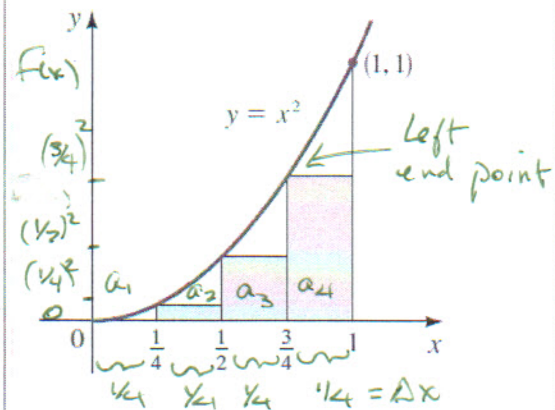
$= \frac{1}{4}\left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16}\right) = \frac{1}{4}\left(\frac{30}{16}\right)$

$R_4 = .46875$ units^2

R indicates the rectangles were created using right end points.

This is an over estimate.

& Subscript of 4 indicates 4 rectangles



$a_1 = 0$

$a_2 = \frac{1}{4}\left(\frac{1}{4}^2\right)$

$a_3 = \frac{1}{4}\left(\frac{1}{2}^2\right)$

$a_4 = \frac{1}{4}\left(\frac{3}{4}^2\right)$

$A_{Total} = \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right)$

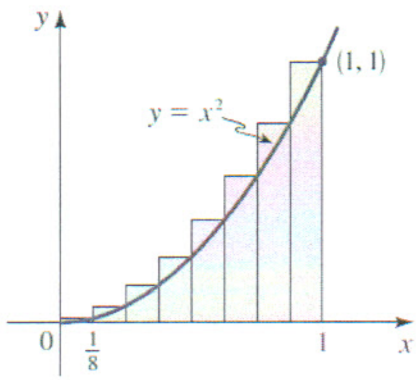
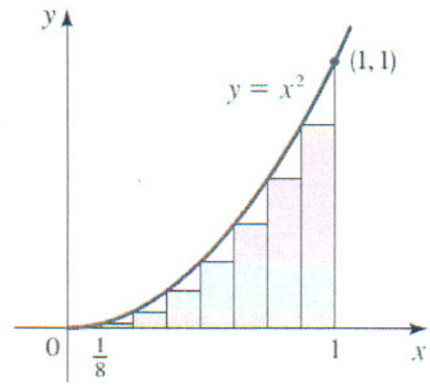
$L_4 = .21875$ units^2

L = left end point used to make 4 rectangles (under estimate)

Actual area:

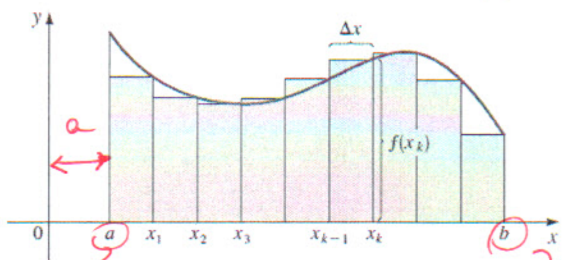
$\boxed{.21875 < A < .46875}$

Same problem....smaller rectangles So... the more rectangles the more accurate the estimate becomes!

Overestimate of area.	Underestimate of area.
 (b) Using right endpoints	 (a) Using left endpoints

As the number of rectangles goes to infinity..... ☺

The area A of the region S that lies under the graph of a continuous function f is the limit of the sum of the areas of the approximating rectangles: ***use right endpoints***.



ending at

AREA=(height) x (width)

$(f(x))(\Delta x)$

n rectangles

starting at "a"

$$A=\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x$$

But "n" goes to infinity!

Δx **is the width** of an approximating rectangle,
x_k **is the right endpoint** of the kth **rectangle**
$f(x_k)$ **is its height**.

$\boldsymbol{n}$ *rectangles*
region from $x=a$ *to* $x=b$

width: $\boldsymbol{\Delta x=\frac{b-a}{n}}$

right endpoint: $\boldsymbol{x_k=a+k\Delta x}$

height: $\boldsymbol{f(x_k)=f(a+k\Delta x)}$

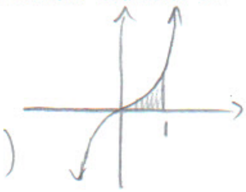
Finding the Area by the Limit Definition

Find the area of the region bounded by the function and the vertical lines x=0 and x=1.

$a \downarrow \quad \downarrow b$

$f(x) = x^3$

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$



$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$= a + k\Delta x$

$x_k = 0 + k\left(\frac{1}{n}\right)$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \left(\frac{1}{n}\right)$$

factor out coefficient

$f(x_k) = f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^3$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3$$

Summation formula

Rearrange

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^4}\,n^2}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{4}\right)\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{4}\right)\left(\underbrace{\frac{\cancel{n}}{\cancel{n}}}_{1} + \cancel{\frac{1}{n}}^{\,0}\right)\left(\frac{\cancel{n}}{\cancel{n}} + \cancel{\frac{1}{n}}^{\,0}\right) = A$$

$$\boxed{A = \frac{1}{4}}$$

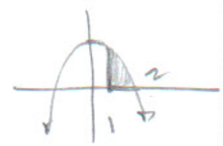
(as $n \to \infty$ $\frac{1}{n} = 0$)

Find the area of the region bounded by the function and the vertical lines x=1 and x=2.

$a \quad b$

$f(x) = 4 - x^2$

$\Delta x = \frac{2-1}{n} = \frac{1}{n}$



$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x)\,\Delta x$$

$x_k = 1 + k\left(\frac{1}{n}\right)$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 - \frac{2k}{n} - \frac{k^2}{n^2}\right)\left(\frac{1}{n}\right)$$

Distribute

$f(x_k) = f\left(1+\frac{k}{n}\right) = 4 - \left(1+\frac{k}{n}\right)^2 \left(1+\frac{k}{n}\right)$

$= 4 - \left(1 + \frac{2k}{n} + \frac{k^2}{n^2}\right) = 3 - \frac{2k}{n} - \frac{k^2}{n^2} = f(x_k)$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n} - \frac{2k}{n^2} - \frac{k^2}{n^3}$$

factor out coefficients & use formulas

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 3 - \frac{1}{n^2}\sum 2k - \frac{1}{n^3}\sum k^2 =$$

$$= \lim_{n\to\infty} \frac{3\cancel{n}}{\cancel{n}} - \frac{\cancel{2}(\cancel{n})(n+1)}{\cancel{2}n\cancel{^2}} - \frac{\cancel{n}(n+1)(2n+1)}{6n^{\cancel{3}\,2}} =$$

$$\lim_{n\to\infty} 3 - 1\left(\underbrace{\frac{\cancel{n}}{\cancel{n}}}_{1} + \cancel{\frac{1}{n}}^{\,0}\right) - \frac{1}{6}\left(\underbrace{\frac{\cancel{n}}{\cancel{n}}}_{1} + \cancel{\frac{1}{n}}^{\,0}\right)\left(\frac{2\cancel{n}}{\cancel{n}} + \cancel{\frac{1}{n}}^{\,0}\right)$$

$$= 3 - 1 - \frac{2}{6} = 2 - \frac{1}{3} = \boxed{\frac{5}{3} = A}$$