Calculus

Lesson 3.4: Concavity and the Second Derivative Test

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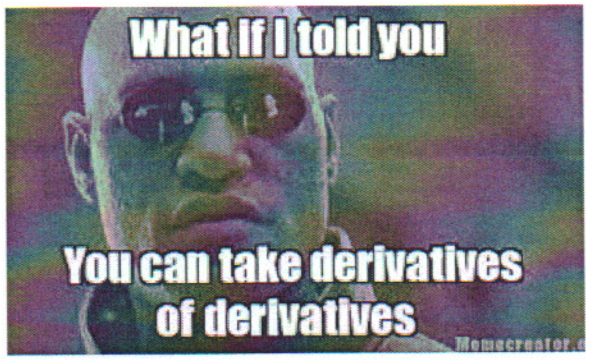
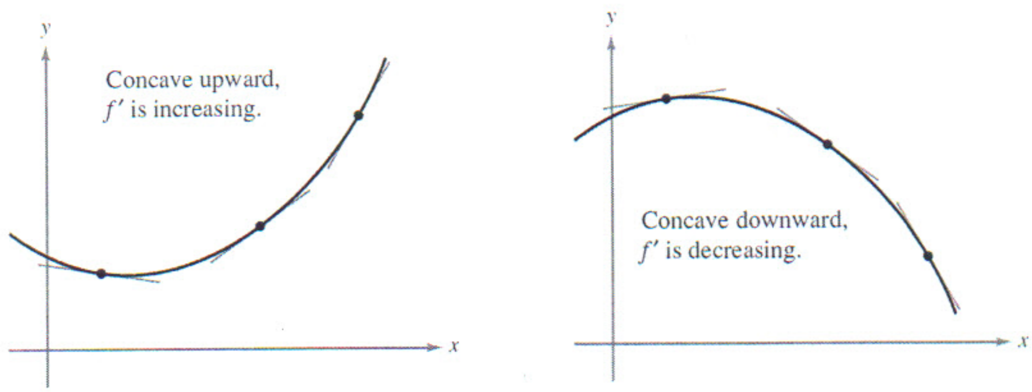


In this section, we will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is curving upward or curving downward.

DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.



(a) The graph of f lies above its tangent lines.

(b) The graph of f lies below its tangent lines.

THEOREM 3.7 TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

To apply this theorem:

1. locate the x-vaues at which
   $f''(x)=0$ or $f''(x)$ *does not exist* (*where the denominator is* $= 0$
2. use these these x-values to determine test intervals
3. test the sign of $f''(x)$ in each of the test intervals

Determining Concavity

Determine the open intervals on which the graph is concave up or concave down.

$$f(x)=\frac{6}{x^2+3}=6\left(x^2+3\right)^{-1}$$

$$f'=-6\left(x^2+3\right)^{-2}(2x)=\frac{-12x}{\left(x^2+3\right)^2}$$

$$f''=\frac{-12\left(x^2+3\right)^{\cancel{2}}-2\left(\cancel{x^2+3}\right)(2x)(-12x)}{\left(x^2+3\right)^{\cancel{4}\,3}}=\frac{-12x^2-36+48x^2}{\left(x^2+3\right)^3}$$

$$=\frac{36x^2-36}{\left(x^2+3\right)^3}=0$$

$36x^2-36=0$

$\cancel{36}\left(x^2-1\right)=0$

$x^2-1=0$

$\underline{x=\pm 1}$ Critical numbers

$x^2+3=0$

$\cancel{x^2=-3}$

-2 0 2

-1 1

x^2-1	+	−	+
x^2+3	+	+	+
	+	−	+

concave: up down up

Ans Concave up: $(-\infty,-1),(1,\infty)$

down: $(-1,1)$

Determine the open intervals on which the graph is concave up or concave down.

$$f(x)=\frac{x^2+1}{x^2-4} \qquad f'=\frac{2x(x^2-4)-2x(x^2+1)}{(x^2-4)^2}$$

$$=\frac{\cancel{2x^3}-8x-\cancel{2x^3}-2x}{(x^2-4)^2}=\frac{-10x}{(x^2-4)^2}$$

$$f''=\frac{-10(x^2-4)^{\cancel{2}}-2\cancel{(x^2-4)}(2x)(-10x)}{(x^2-4)^{\cancel{4}\,3}}=\frac{-10x^2+40+40x^2}{(x^2-4)^3}$$

$$=\frac{30x^2+40}{(x^2-4)^3}=\frac{\cancel{10}(3x^2+4)}{(x^2-4)^3}=0$$

$3x^2+4=0$

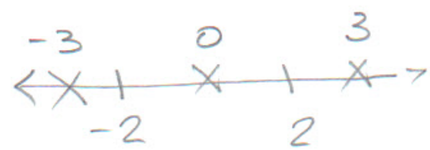
$\cancel{x^2=\frac{-4}{3}}$

no pts

$x^2-4=0$

$x^2=4$

$\underline{x=\pm 2}$ Critical #'s



$3x^2+4$	+	+	+
x^2-4	+	−	+
	+	−	+
concave	up	down	up

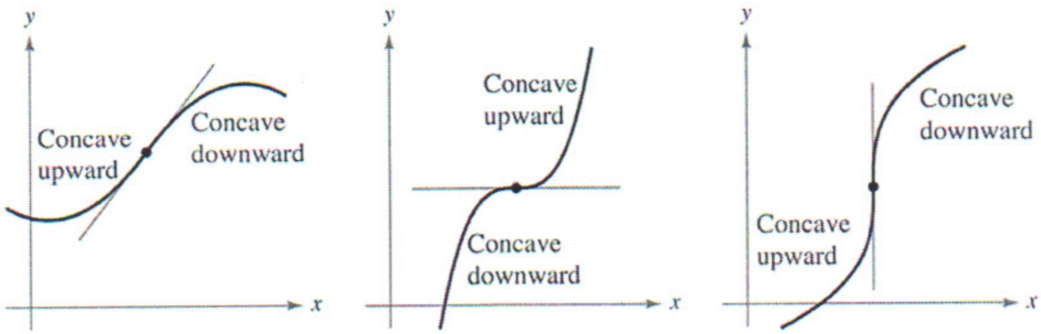
Ans: concave up $(-\infty,-2)(2,\infty)$

concave down $(-2,2)$

DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

The point where the concavity changes and the tangent line to the graph exists, is a **point of inflection.**



The concavity of f changes at a point of inflection. Note that a graph crosses its tangent line at a point of inflection.

To locate possible points of inflection, you can determine the values of x where $f''(x)=0$ or $f''(x)$ *does not exist*. The process is similar to locating extrema of f.

THEOREM 3.8 POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.

Note, the converse of this theorem is not necessarily true! Think of the parent quadratic function. $y = x^2$ is concave upwards from $-\infty < x < 0$ *and* $0 < x < \infty$, however its second derivative is 0 at $x = 0$ but, (0,0) is not a point of inflection.

Determine the points of inflection and discuss the concavity of the graph.

$f(x) = x^4 - 4x^3$ $\longrightarrow$ 2nd derivative = 0

$f' = 4x^3 - 12x^2$

$f'' = 12x^2 - 24x$

$12x(x-2) = 0$

$x = 0 \quad x = 2 \Leftarrow$ points of inflection

	-1	0	1	2	3
$12x$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
	$+$		$-$		$+$
	concave up		down		up

Ans

Concave up: $(-\infty, 0), (2, \infty)$

concave down: $(0, 2)$

The seond derivative test may beused to preform a simple test for relative maxima and minima along with testing for concavity.

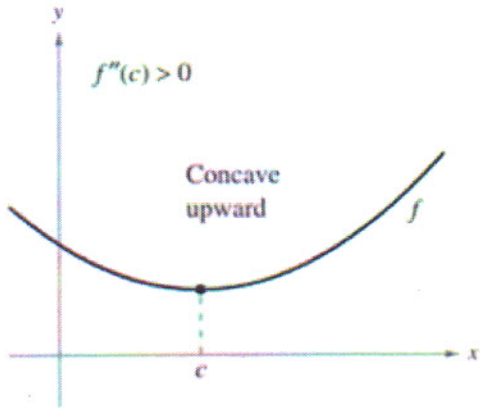
THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

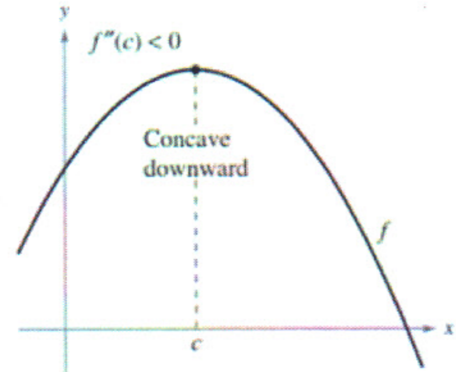
1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

- Locate critical numbers These are where we may have a minimum or maximum or neither
- If f'' is positive, the function is concave up and c is a minimum
- If f'' is negative, the function is concave down and c is a maximum.
- If $f''=0$, c is neither minimum or maximum



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Find the relative extrema.

$f(x) = -3x^5 + 5x^3$

$f' = -15x^4 + 15x^2$

$-15x^2(x^2 - 1) = 0$

$x = 0, \pm 1$ critical numbers

$f'' = -60x^3 + 30x$

* $f''(0) = 0$ fails neither min nor max

$f''(1) = -60 + 30 = -30$ relative max at $x = 1$

$f''(-1) = 60 - 30 = 30$ relative min at $x = -1$

* Note First derivative test will show that f increases to the left and right of $x = 0$