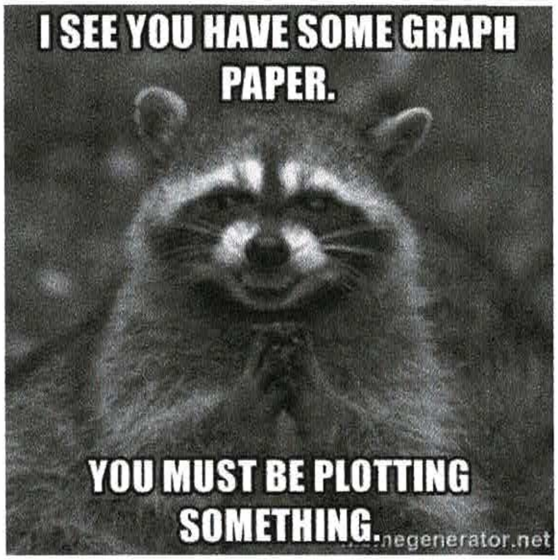


Calculus
Lesson 3.6: A Summary of Curve Sketching
Mrs. Snow, Instructor

In precalclulus we analyzed and sketched graphs of functions. While we determined x and y-intercepts, asymptotes, and the function's behavior as it approached the asymptotes, there is more that we can calculate using the derivatives of the function.

Steps in Analyzing the Graph of a Function

a. X-intercepts, Y-intercepts:
b. Vertical Asymptotes, Horizontal Asymptotes:
c. First derivative: Critical Points:
d. Increasing interval, Decreasing interval:
e. Second Derivative: Inflection Points:
f. Concave Up: Concave Down:
g. Sketch the Graph:

1. Analyze and sketch the graph of $f(x) = \frac{2(x^2-9)}{x^2-4}$ $= \frac{2x^2-18}{x^2-4} = f(x)$

a. X-intercepts, Y-intercepts:

b. Vertical Asymptotes, Horizontal Asymptotes:

c. First derivative: Critical Points:

d. Increasing interval, Decreasing interval:

e. Second Derivative: Inflection Points:

f. Concave Up: Concave Down:

g. Sketch the Graph:

$$f'(x) = \frac{4x(x^2-4) - 2x(2x^2-18)}{(x^2-4)^2}$$

$$= \frac{\cancel{4x^3} - 16x - \cancel{4x^3} + 36x}{(x^2-4)^2} = \boxed{\frac{20x}{(x^2-4)^2} = f'}$$

$$f''(x) = \frac{20(x^2-4)^{\cancel{2}} - 2\cancel{(x^2-4)}(2x)(20x)}{(x^2-4)^{\cancel{4}\,3}} = \frac{20(x^2-4) - 80x^2}{(x^2-4)^3} =$$

$$= \frac{20x^2 - 80 - 80x^2}{(x^2-4)^3} = \frac{-60x^2-80}{(x^2-4)^3} = \boxed{\frac{-20(3x^2+4)}{(x^2-4)^3} = f''}$$

a/b Intercepts $(X,0)(0,y) = \boxed{(3,0)(-3,0)\ (0,\frac{9}{2})}$

$2(x^2-9) = 0$

$x^2-9 = 0$

$(x+3)(x-3) = 0$

$x = \pm 3$

$\frac{2(0^2)-18}{0^2-4} = y$

$\frac{18}{4} = \frac{9}{2} = y$

VA & HA

VA:

$x^2-4 = 0$

$x = \pm 2$

HA:

BOBO

BOTN

EATSDC

EATSDC

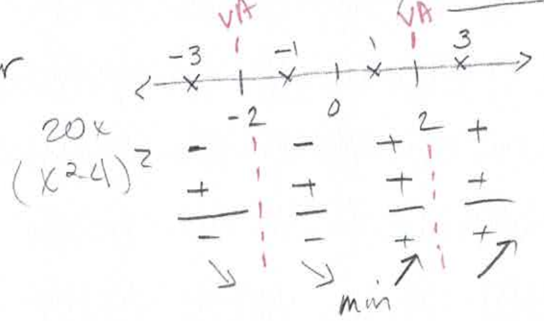
$x = 2, x = -2$ $y = 2$

c/d 1st der, Incr/decr

$f' = \frac{20x}{(x^2-4)^2} = 0$

$20x = 0$ $x^2-4 = 0$

$x = 0$ $x = \pm 2$



min $(0, \frac{9}{2})$

decr $(-\infty,-2)(-2,0)$

incr $(0,2)(2,\infty)$

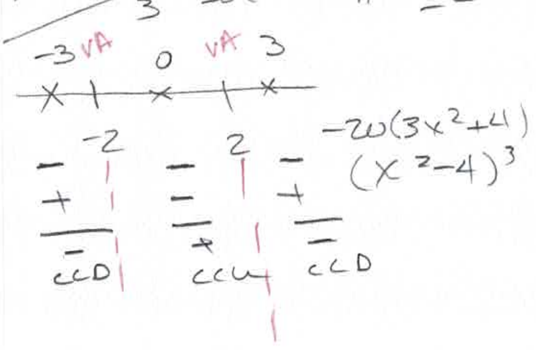
e/f concavity

$-\cancel{20}(3x^2+4) = 0$

~~$x^2 = \frac{-4}{3}$ no Real sol~~

$(x^2-4)^3 = 0$

$x = \pm 2$

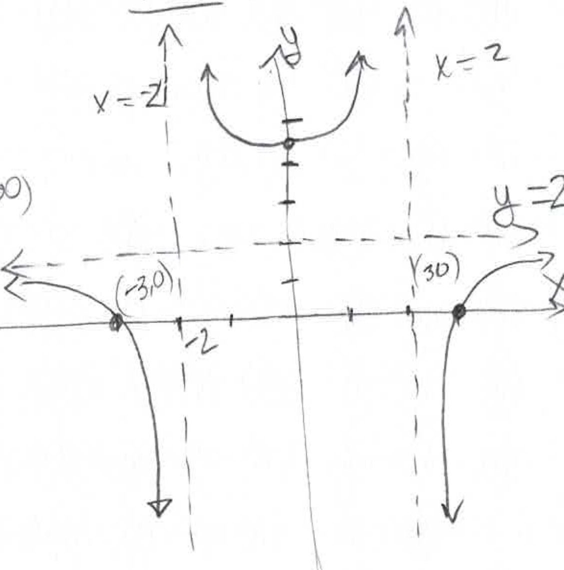


CCD

$(-\infty,-2)(2,\infty)$

CCU

$(-2,2)$



2. Analyze and sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$

a. X-intercepts, Y-intercepts:
b. Vertical Asymptotes, Horizontal Asymptotes:
c. First derivative: Critical Points:
d. Increasing interval, Decreasing interval:
e. Second Derivative: Inflection Points:
f. Concave Up: Concave Down:
g. Sketch the Graph:

$$f' = \frac{(2x-2)(x-2) - 1(x^2-2x+4)}{(x-2)^2}$$

$$= \frac{\cancel{2x^2} - 6x + \textcircled{4} - \cancel{x^2} + 2x \textcircled{+4}}{(x-2)^2}$$

$$\boxed{f' = \frac{x^2-4x}{(x-2)^2}}$$

$$f' = \frac{(2x-4)(x-2)^{\cancel{2}} - 2\cancel{(x-2)}(x^2-4x)}{(x-2)^{\cancel{4}\,3}}$$

$$= \frac{\cancel{2x^2} - \cancel{8x} + 8 - \cancel{2x^2} + \cancel{8x}}{(x-2)^3} = \boxed{\frac{8}{(x-2)^3} = f''}$$

$\boxed{a/B}$ Incpt $(x,0)$ $(0,y)$ $\boxed{(0,-2)}$

$x^2-2x+4=0$? use discriminant
$4-4(1)(4)$
$=$ neg $\Rightarrow$ no real sol

$$\frac{0^2-2(0)+4}{0-2} = -2$$

VA $x=2$

HA BOTN $\Rightarrow$ None But Oblique Asym.

$\boxed{c/D}$ Incr/Decr

$x^2-4x=0$ $(x-2)^2=0$
$x(x-4)=0$ $x=2$
$x=0,\ x=4$

Test points: −1, 1, 3, 5; marks: 0, 2, 4

x	−	+	+	+
$x-4$	−	−	−	+
$(x-2)^2$	+	+	+	+
	+	−	−	+
	↗ max	↘ VA	↘ min	↗

Incr $(-\infty,0)$ $(4,\infty)$
Decr $(0,2)$ $(2,4)$
max $(0,-2)$
min $(4,6)$

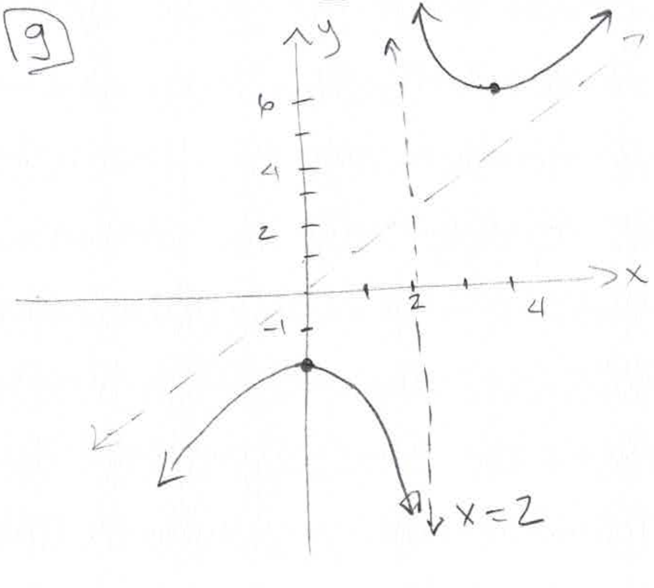
$\boxed{e/F}$ Infl pt $(x-2)^3=0$
$x=2$

Test points: 0, 3; mark: 2

8	+	+
$(x-2)^3$	−	+
	−	+
	CCD $(-\infty,2)$	CCU $(2,\infty)$

VA

$\boxed{g}$



3. Analyze and sketch the graph of $f(x) = x\sqrt{9 - x^2}$ $= x(9-x^2)^{1/2}$ Domain $-3 \leq x \leq 3$ *

a. X-intercepts, Y-intercepts:

b. Vertical Asymptotes, Horizontal Asymptotes:

c. First derivative: Critical Points:

d. Increasing interval, Decreasing interval:

e. Second Derivative: Inflection Points:

f. Concave Up: Concave Down:

g. Sketch the Graph:

$$f' = (1)(9-x^2)^{1/2} + \frac{1}{\cancel{2}}(9-x^2)^{-1/2}(-\cancel{2}x)(x)$$

$$= (9-x^2)^{-1/2}\left[9-x^2-x^2\right] \boxed{= \frac{9-2x^2}{(9-x^2)^{1/2}} = f'}$$

$$f'' = \frac{-4x\,(9-x^2)^{1/2} \ominus \cancel{\tfrac{1}{2}}(9-x^2)^{-1/2}(\overset{+}{\ominus}\cancel{2}x)(9-2x^2)}{(9-x^2)^1}$$

$$f'' = \frac{(9-x^2)^{-1/2}\left[-4x(9-x^2)^1 + 9x - 2x^3\right]}{(9-x^2)^1} = \frac{-36x+4x^3+9x-2x^3}{(9-x^2)^{3/2}} =$$

$$= \frac{2x^3-27x}{(9-x^2)^{3/2}} = \boxed{\frac{x(2x^2-27)}{(9-x^2)^{3/2}} = f''}$$

$\boxed{a/b}$ Intercepts (0,0) (±3,0) VA/HA - NONE

*Domain restrictions: $\sqrt{9-x^2} \Rightarrow [-3, 3]$

$9-x^2 > 0$

$x^2 < 9$

$x < \pm 3$

$\boxed{c/D}$ min/MAX

$9-2x^2 = 0$

$9 = 2x^2$

$x^2 = \frac{9}{2}$

$x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$

$\sim \pm 2.1$

		2.5		0		2.5
	-3		-2.1		2.1	3
$9-2x^2$		$-$		$+$		$-$
$(9-x^2)^{1/2}$		$+$		$+$		$+$
		$-$		$+$		$-$
		↘	min	↗	max	↘

$\left(-\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$ $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$

$\boxed{e/F}$ concavity

$x(2x^2-27) = 0$

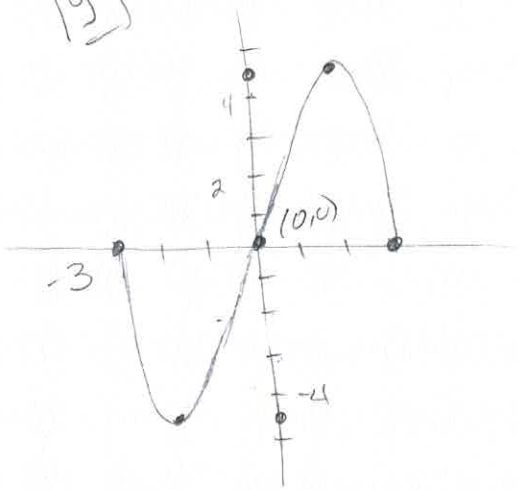
$x = 0$ $\cancel{x^2 = \frac{27}{2}}$ Domain

	-1	0	1
x	$-$		$+$
$2x^2-27$	$-$		$-$
$(9-x^2)^{3/2}$	$+$		$+$
	CCU		CCD

$(-\infty, 0)$ $(0, \infty)$

CCU CCD

$\boxed{g}$



4. Analyze and sketch the graph of $f(x) = x^4 - 4x^3$

a. X-intercepts, Y-intercepts:

b. Vertical Asymptotes, Horizontal Asymptotes:

c. First derivative: Critical Points:

d. Increasing interval, Decreasing interval:

e. Second Derivative: Inflection Points:

f. Concave Up: Concave Down:

g. Sketch the Graph:

$f' = 4x^3 - 12x^2 = 4x^2(x-3) = f'$

$f'' = 12x^2 - 24x = 12x(x-2) = f''$

a/B

$x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

$x = 0, x = 4$

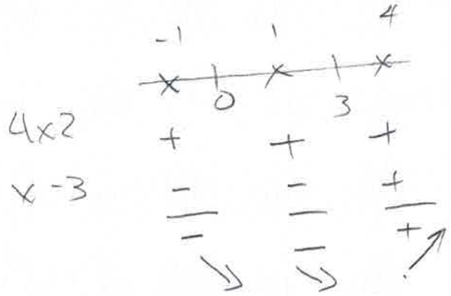
(0,0)
(4,0) Intercepts

NO VA/HA

c/D min max

$4x^2(x-3) = 0$

$x = 0 \quad x = 3$



min
(3,27)

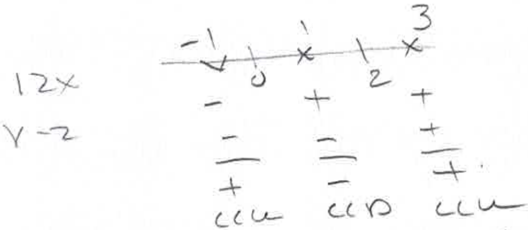
decr $(-\infty, 3)$

incr $(3, \infty)$

e/f

$12x(x-2) = 0$

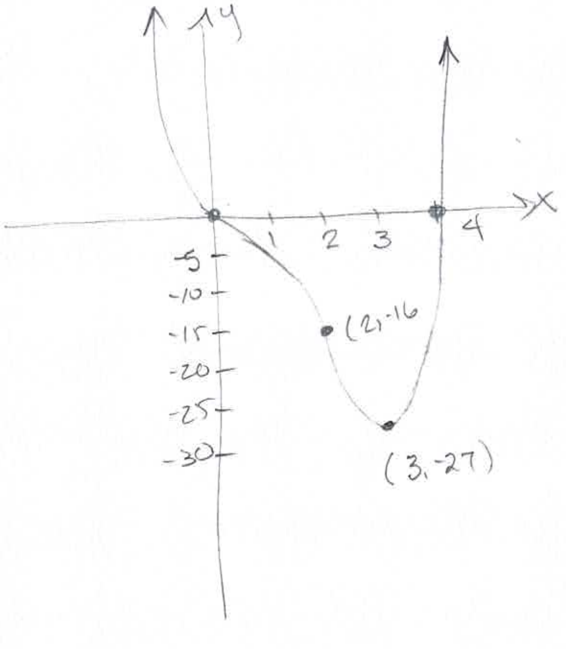
$x = 0, \quad x = 2$



$(-\infty, 0)$ $(0, 2)$ $(2, \infty)$

Infl pt (0,0) (2,-16)

g



5. Analyze and sketch the graph of $f(x) = \sin x - \sqrt{3}\cos x$ *for the interval* $[0, 2\pi]$

a. X-intercepts, Y-intercepts:

b. Vertical Asymptotes, Horizontal Asymptotes: $f' = \cos x + \sqrt{3}\sin x$

c. First derivative: Critical Points:

d. Increasing interval, Decreasing interval: $f'' = -\sin x + \sqrt{3}\cos x$

e. Second Derivative: Inflection Points:

f. Concave Up: Concave Down:

g. Sketch the Graph:

a/b

$y = \cancel{\sin 0} - \sqrt{3}\cos 0$

$y = -\sqrt{3}$

$\sin x = \sqrt{3}\cos x$

$\frac{\sin x}{\cos x} = \tan x = \sqrt{3}$

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

$(0, -\sqrt{3})$ $(\frac{\pi}{3}, 0)$

$(\frac{4\pi}{3}, 0)$

NO - VA / HA

c/D

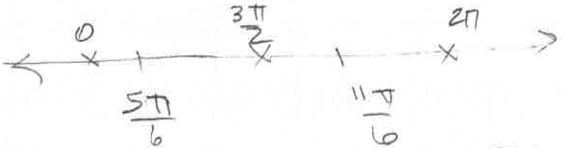
min/Max

$\cos x + \sqrt{3}\sin x = 0$

$\sqrt{3}\sin x = -\cos x$

$\tan x \; \frac{\sin x}{\cos x} = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

$x = \frac{5\pi}{6}, \frac{11\pi}{6}$



$1 + \sqrt{3}(0)$ $0 + \sqrt{3}(-1)$ $1 + \sqrt{3}(0)$

$+$ $-$ $+$

max min

$(\frac{5\pi}{6}, 2)$ $(\frac{11\pi}{6}, -2)$

Incr $(0, \frac{5\pi}{6})$, $(\frac{11\pi}{6}, 2\pi)$

decr $(\frac{5\pi}{6}, \frac{11\pi}{6})$

e/f

concavity

$-\sin x + \sqrt{3}\cos x = 0$

$\sin x = \sqrt{3}\cos x$

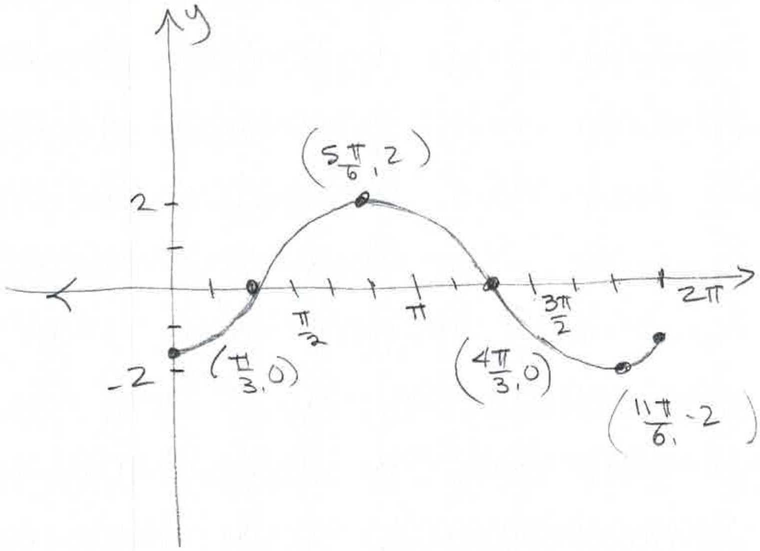
$\tan x = \frac{\sin x}{\cos x} = \sqrt{3}$

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

$0 + \sqrt{3}(1)$ $0 + \sqrt{3}(-1)$ $0 + \sqrt{3}(1)$

$+$ $-$ $+$

CCU CCD CCU



CCU $[0, \frac{\pi}{3})$ $(\frac{4\pi}{3}, 2\pi]$

CCD $(\frac{\pi}{3}, \frac{4\pi}{3})$

Inflpt $(\frac{\pi}{3}, 0)$ $(\frac{4\pi}{3}, 0)$