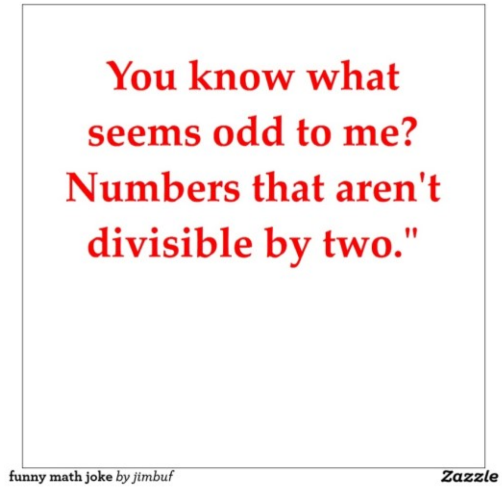
Precalculus
Lesson 12.3 Geometric Sequences
Mrs. Snow, Instructor



Another type of sequence is the geometric sequence. It occurs in applications to finance and population growth. An arithmetic sequence, we found that we added a number to the initial term to form the sequence. In geometric sequences, we start with a number and then generate a sequence by repeatedly multiplying a nonzero number r.

A **geometric sequence*** may be defined recursively as $a_1 = a$, $\frac{a_n}{a_{n-1}} = r$, or as

$$\boxed{a_1 = a, \qquad a_n = ra_{n-1}}$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is the first term, and the nonzero number r is called the **common ratio.**

Identify the first term and common ratio: $2, 6, 18, 54, 162, \ldots$
Identify the first term and common ratio: $\{s_n\} = 2^{-n}$

Press pause and take a moment work the examples; press play when ready to move on

Identify the first term and common ratio:

2, 6, 18, 54, 162, ...

$a_1 = 2$

$r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18}$

$r = 3$

Identify the first term and common ratio:

$\{s_n\} = 2^{-n}$

$S_1 = 2^{-1} = \frac{1}{2}$

$S_2 = 2^{-2} = \frac{1}{4}$

$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4}\left(\frac{2}{1}\right) = \frac{1}{2} = r$

Identify the first term and common ratio:

$\{t_n\} = \{3 \cdot 4^n\}$

Press pause and take a moment work the examples; press play when ready to move on

Identify the first term and common ratio:

$\{t_n\} = \{3 \cdot 4^n\}$

$t_1 = 3(4^1) = 12$

$r = \frac{48}{12} = 4$

$t_2 = 3(16) = 48$

Find a Formula for a Geometric Sequence

*n*th Term of a Geometric Sequence

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula

$$a_n = a_1 r^{n-1} \qquad r \neq 0 \qquad \textbf{(2)}$$

to find the nth term, we need to know the first term, a_1, and the common ratio.

Given the sequence $10, 9, \frac{81}{10}, \frac{729}{100}, \ldots$

a) Find the nth term
b) find the 9th term
c) find a recursive formula for the sequence

Press pause and take a moment work the examples; press play when ready to move on

Given the sequence $10, 9, \frac{81}{10}, \frac{729}{100}, \ldots$

$r = \frac{9}{10}$

a) Find the nth term
b) find the 9th term
c) find a recursive formula for the sequence

(a) $a_n = 10\left(\frac{9}{10}\right)^{n-1}$

(b) $a_9 = 10\left(\frac{9}{10}\right)^{9-1} = 10\left(\frac{9}{10}\right)^{8} = 4.3046721$

note! this is exact as calculator stops here!

(c) per definition:

$a_1 = 10 \quad a_n = \frac{9}{10}(a_{n-1})$

Find the sum of the first n terms of a geometric sequence, (Partial Sum)

Given a geometric sequence, we can calculate the sum of any given number of the terms.

Sum of the First *n* Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r, where $r \neq 0, r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} = \sum_{k=1}^{n} a_1 r^{k-1}$$

$$= a_1 \cdot \frac{1 - r^n}{1 - r} \qquad r \neq 0, 1$$

KNOW THIS FORMULA!!!

Find the sum S_n, for the first n terms of the geometric series:

$\left(\frac{1}{2}\right)^n$

Press pause and take a moment work the examples; press play when ready to move on

Find the sum S_n, for the first n terms of the geometric series:

$\left(\frac{1}{2}\right)^n$

$S_n = \frac{1}{2} \cdot \frac{1 - \frac{1}{2}^n}{1 - \frac{1}{2}} = \frac{\cancel{1}}{\cancel{2}} \cdot \frac{1 - \frac{1}{2}^n}{\cancel{\frac{1}{2}}}$

$S_n = 1 - \frac{1}{2}^n$

$S_1 = \frac{1}{2}$

$S_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4}\left(\frac{2}{1}\right) = \frac{1}{2}$

If we think about a geometric series, we will realize that it will continue to go on and on. Hence it is called and **Infinite Geometric Series**.

If we set about to find the sum of an infinite geometric series, it will do one of two things. First the sum S_n will approach a specific value or it **converges.** If not, then it is called a **divergent.**
divergent example: 1, 2, 4, 8, 16, 32, 64, 128, 256, ...
convergent example: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \ldots.$

Convergence of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}}$$

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Note: this is the sum of the whole geometric series

Determine if the geometric series converges or diverges. If it converges, find its sum.

$$\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1}$$

Press pause and take a moment work the examples; press play when ready to move on

Determine if the geometric series converges or diverges. If it converges, find its sum.

$$\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1} = \frac{2}{1-\frac{2}{3}}$$

converges ?

is $|r| < 1$?

$\left|\frac{2}{3}\right| < 1$ yes convergent!

$$= \frac{2}{\frac{1}{3}}$$

$a_1 = 2\left(\frac{2}{3}\right)^{1-1} = 2$

$$= \frac{2}{1}\left(\frac{3}{1}\right) = \underline{\underline{6}}$$

$r = \frac{2}{3}$

Writing a Repeated Decimal as a Fraction:
Show that the repeating decimal $0.999\ldots = 1$

Press pause and take a moment work the examples; press play when ready to move on

Writing a Repeated Decimal as a Fraction:
Show that the repeating decimal 0.999 … = 1

Since we are talking About sums of infinite series:

$.999... = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$

$.9 + .09 + .009 ...$

$r = \frac{\frac{9}{100}}{\frac{9}{10}} = \frac{9}{100}\left(\frac{10}{9}\right)$

$r = \frac{1}{10}$

$\left|\frac{1}{10}\right| < 1$

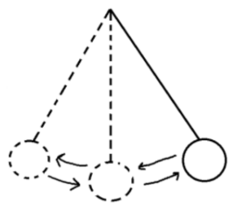
$\sum_{K=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{K-1} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$

QED :)

So converses to 1

Initially, a pendulum swings through an arc of 18 inches. On each successive swing, the length of the arc is 0.98 of the previous length.

a) What is the length of the arc of the 10^{th} swing?
b) On which swing is the length of the arc first less than 12 inches?
c) After 15 swings, what total distance will the pendulum have swung?
d) When it stops, what total distance will the pendulum have swung?



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d) When it stops, what total distance will the pendulum have swung?

(a) 1st swing = 18 in

2nd swing = 18(.98)

3rd = [(18)(.98)] .98

so: $a_1 = 18$

$r = .98$

$a_{10} = a_1(r^9) = 18(.98)^9 = \underline{15.007 \text{ inches}}$

(b) $18(.98^{n-1}) = 12$

$\ln .98^{n-1} = \ln \frac{12}{18}$

$(n-1)\ln .98 = \ln 12/18$

$n = \frac{\ln 12/18}{\ln .98} + 1$

$n = 21.07$

∴ 22nd swing pendulum goes below 12 inches

(c) Sum of 15 swings

$S_{15} = 18\left(\frac{1-.98^{15}}{1-.98}\right) =$

$\approx \underline{235.3 \text{ inches}}$

(d) total distance? $|r|<1$

so sum of infinite series

$\sum_{k=1}^{\infty} 18(.98^{k-1}) = \frac{18}{1-.98} = \underline{900 \text{ in}}$