Calculus

Lesson 5.5: Bases other Than e and Applications

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No, I got bored first

and invented Calculus

The base of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

DEFINITION OF EXPONENTIAL FUNCTION TO BASE a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Half-life is the time it takes for half of the material to decay. When modeling the half-life of a radioactive substance, it is convenient to use ½ as the base for the exponential model:

$$A = A_0 \frac{1^{\frac{t}{k}}}{2}$$

A = amount of remaining material; A_0 = initial amount;, decay factor=$1 - r$= ½ ; t=time; k=half-life time

Radioactive Half-Life Model

- The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbo-14. How much will be present in 10000 years?

$$A = 1\left(\frac{1}{2}\right)^{10000/5715}$$

$$= .3 \text{ grams}$$

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

DEFINITION OF LOGARITHMIC FUNCTION TO BASE a

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

(change of base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$)

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other. Remember that the common log is base 10 or $\log x = \log_{10} x$

PROPERTIES OF INVERSE FUNCTIONS

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all x

Bases Other Than e Solve for x in each equation.

a. $3^x = \frac{1}{81} = \frac{1}{3^4}$

$3^x = 3^{-4}$

$\boxed{x = -4}$

b. $\log_2 x = -4$

$2^{-4} = x = \boxed{1/16}$

THEOREM 5.13 DERIVATIVES FOR BASES OTHER THAN e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

Differentiating Functions to Other Bases

- Find the derivative of each function.

a. $y = 2^x$

$= (\ln 2)(2^x)$

b. $y = 2^{3x}$

$= \ln 2 \cdot (2^{3x}) \frac{d}{dx}(3x)$

$= 3 \ln 2 (2^{3x})$

c. $y = \log_{10} \cos x =$

$\frac{1}{(\ln 10)(\cos x)} \frac{d}{dx}(\cos x)$

$= \frac{1}{(\ln 10)(\cos x)}(-\sin x)$ tan x

$= \frac{-1}{\ln 10} \cdot \tan x$

Integrals for Bases Other than e

$$\int a^x dx = \left(\frac{1}{\ln a}\right)a^x + C$$

$$\int 2^x dx = \frac{1}{\ln 2} \cdot 2^x + C$$

THEOREM 5.14 THE POWER RULE FOR REAL EXPONENTS

Let n be any real number and let u be a differentiable function of x.

1. $\frac{d}{dx}[x^n] = nx^{n-1}$
2. $\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

Comparing Variables and Constants

$$\frac{d}{dx}[e^e] = 0 \quad (e^e \text{ is a constant})$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[x^e] = e \cdot x^{e-1}$$

$y = x^x$

$\ln y = x \ln x$

$(y)\frac{y'}{y} = (1)\ln x + x\left(\frac{1}{x}\right)(y)$

$y' = (\ln x + 1)(x^x)$

Interest Formulas

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(t) = Pe^{rt}$$

- A(t)= amount after t years
- P=Principal
- r=interest rate per year
- n=number of times compounded per year
- t=number of years

Comparing Continuous and Quarterly Compounding

- A deposit of \$2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the account at the end of 5 years if the interest is compounded a) quarterly, b) monthly, and c) continuously.

a) $n = 4$

$$A = 2500\left(1 + \frac{.05}{4}\right)^{4(5)}$$

$\sim \$3205.09$

b) $n = 12$

$$A = 2500\left(1 + \frac{.05}{12}\right)^{12(5)}$$

$\sim \$3208.40$

c) @ e

$$A = 2500\, e^{.05(5)}$$

$= \$3210.06$

Bacterial Culture Growth

- A bacterial culture is growing according to the logistic growth function

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}} \quad t \geq 0$$

where y is the weight of the culture in grams and t is the time in hours.

Find the weight of the culture after 1 hour. Find the rate at which y is changing at t=1 hours and t=10 hours.

$t = 1$ $\quad y = \frac{1.25}{1 + .25e^{-.4(1)}}$ $\quad y = 1.071$ grams.

rate = y': $\quad y' = \frac{-1.25(.25)(-.4)e^{-.4t}}{(1 + .25e^{-.4t})^2} = \frac{.125e^{-.4t}}{(1 + .25e^{-.4t})^2}$

at $t = 1$ $\quad y' = .0615$ g/h

at $t = 10$ $\quad y' = .0023$ g/hr