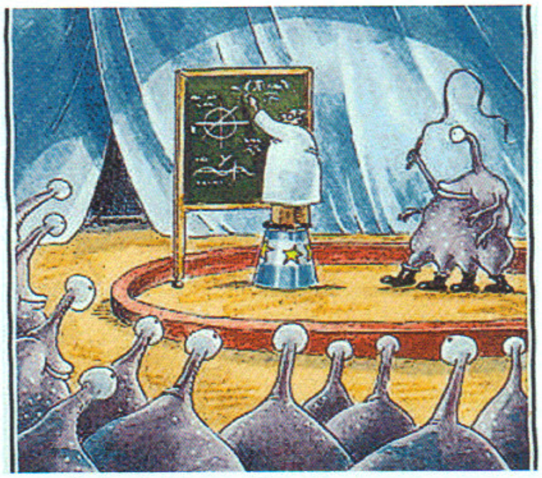
Calculus
Lesson 4.6: Numerical integration
Mrs. Snow, Instructor



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

Some functions simply do not have antiderivatives are elementary functions. In these cases, the Fundamental Theorem of Calculus cannot be applied and we need to use an approximation technique.

One technique, Trapezoidal Rule, can help us to approximate a definite integral by using breaking the area of the region defined by the definite integral into n trapezoids.

THEOREM 4.17 THE TRAPEZOIDAL RULE

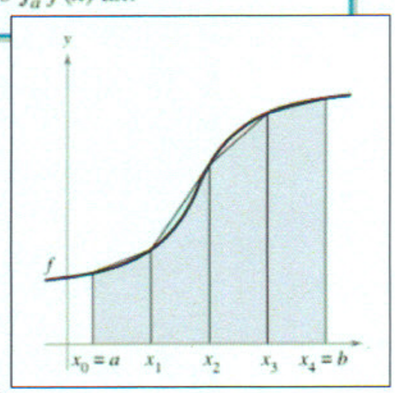
Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\,dx$ is given by

Coefficients: 1 2 2 2..... 2 2 1

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

Partition the interval [a, b] into n subintervals, each of width $\Delta x = \frac{b-a}{n}$



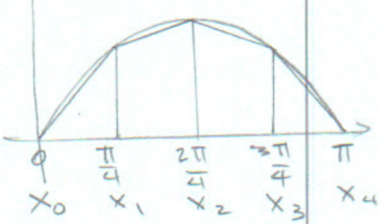
Approximation with the Trapezoid Rule

$n=4: \Delta x = \frac{\pi-0}{4} = \frac{\pi}{4}$

Use the Trapezoid Rule to approximate the following. Use n=4 and n=8.

$$\int_0^{\pi} \sin x \, dx$$

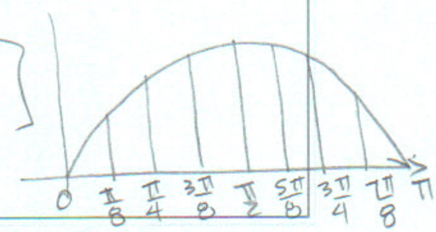
$n=4$:



$$\frac{\pi-0}{2(4)}\left[\underset{f(x_0)}{1\cancel{\sin} 0} + \underset{f(x_1)}{2\sin\frac{\pi}{4}} + \underset{f(x_2)}{2\sin\frac{2\pi}{4}} + \underset{f(x_3)}{2\sin\frac{3\pi}{4}} + \underset{f(x_4)}{\cancel{\sin}\frac{4\pi}{4}}\right]$$

$$= \frac{\pi}{8}\left[2\left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4}\right)\right] =$$

$$= \frac{\pi}{8}(4.8284) \approx \boxed{1.896}$$



$$\frac{\pi}{16}\left[\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{2\pi}{8} + 2\sin\frac{3\pi}{8} + 2\sin\frac{4\pi}{8} + 2\sin\frac{5\pi}{8} + 2\sin\frac{6\pi}{8} + 2\sin\frac{7\pi}{8} + \sin\pi\right]$$

$$= \frac{\pi}{16}(10.0547) = \boxed{1.974}$$

Simpson's Rule: For Simpson's Rule we approximate the function using parabolic approximations. The parabolas intersect with points on the subintervals of the given function. To develop Simpson's Rule for approximating a definite integral, you again partition the interval [a, b] into n subintervals, each of width $\Delta x = (b-a)/n$, however this time we need to have **n be even.**

Note: $\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -\cos\pi - (-\cos 0) = 1 + 1 = \boxed{2}$

THEOREM 4.19 SIMPSON'S RULE

Let f be continuous on $[a, b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)\,dx$ is

Coefficients: 1, 4, 2, 4, 2 ..... 4, 2, 4, 2, 4, 1

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[1f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + 1f(x_n)].$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.

Approximation with Simpson's Rule

$n=4$, $\Delta x = \frac{\pi}{4}$ $n=8$, $\Delta x = \frac{\pi}{8}$

Use Simpson's Rule to approximate the following. Use n=4 and n=8.

$$\int_0^{\pi} \sin x \, dx$$

$n=4$:

$$\frac{\pi}{3(4)}\left[\cancel{\sin} 0 + 4\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 4\sin\frac{3\pi}{4} + \cancel{\sin}\pi\right] = \frac{\pi}{12}(7.6569) = \boxed{2.005}$$

$n=8$

$$\frac{\pi}{3(8)}\left[\sin 0 + 4\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 4\sin\frac{3\pi}{8} + 2\sin\frac{\pi}{2} + 4\sin\frac{5\pi}{8} + 2\sin\frac{3\pi}{4} + 4\sin\frac{7\pi}{8} + \sin\pi\right]$$

$$= \frac{\pi}{24}(15.2809) = \boxed{2.0003}$$