Calculus
Lesson 5.1: The Natural Logarithmic Function: Differentiation
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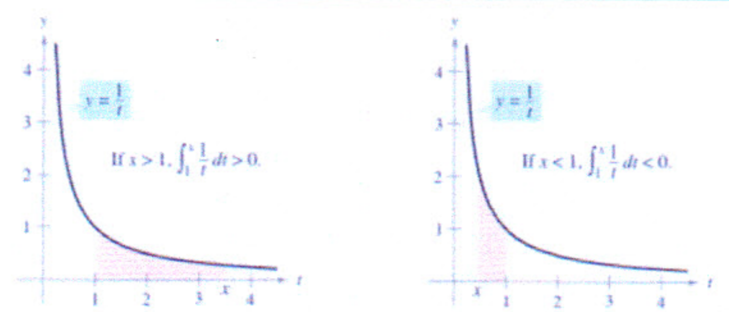
When we were introduced to the General Power Rule, it came with an important disclaimer – it does not apply when $n = -1$. So, what is the antiderivative of $f(x) = \frac{1}{x}$??? Well, that is where the Second Fundamental Theorem of Calculus comes in to play; it will allow us to define this crazy function!

DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.



If $x > 1$, then $\ln x > 0$.

If $0 < x < 1$, then $\ln x < 0$.

THEOREM 5.1 PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

THEOREM 5.2 LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Expand the following logarithms

a. $\ln\frac{10}{9} = \ln 10 - \ln 9$

b. $\ln\sqrt{3x+2} = \ln(3x+2)^{1/2} = \frac{1}{2}\ln(3x+2)$

c. $\ln\frac{6x}{5} = \ln 6x - \ln 5 = \ln 6 + \ln x - \ln 5$

expand too

distribute

d. $\ln\frac{(x^2+3)^2}{\left(x\sqrt[3]{x^2+1}\right)} = \ln(x^2+3)^2 - \left(\ln x\,(x^2+1)^{1/3}\right)$

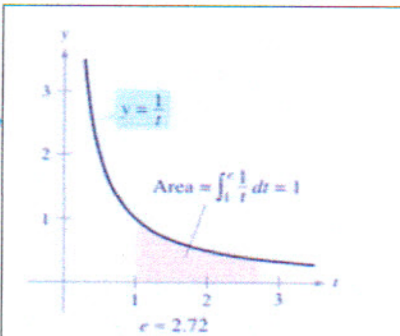
$= 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$

The Number e

DEFINITION OF e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$



e is the base for the natural logarithm because $\ln e = 1$.

Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

calculator – plug & chug!

THEOREM 5.3 DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

Differentiation of Logarithmic Functions

a. $\frac{d}{dx}[\ln(2x)] = \frac{1}{u}\frac{du}{dx} = \frac{1}{2x}\frac{d}{dx}2x = \frac{\cancel{2}}{\cancel{2}x} = \boxed{\frac{1}{x}}$

$u = 2x$

b. $\frac{d}{dx}[\ln(x^2 + 1)]$ $\frac{1}{u}\frac{du}{dx} = \frac{1}{x^2+1}\frac{d}{dx}x^2+1 = \frac{1}{x^2+1}(2x) = \boxed{\frac{2x}{x^2+1}}$

$u = x^2+1$

c. $\frac{d}{dx}[x \ln x] = \frac{d}{dx}x(\ln x) + x\frac{d}{dx}\ln x = (1)\ln x + x\left(\frac{1}{x}\right) = \boxed{\ln x + 1}$

Product rule

d. $\frac{d}{dx}[(\ln x)^3] = 3(\ln x)^2\frac{d}{dx}\ln x = \boxed{3(\ln x)^2\left(\frac{1}{x}\right)}$

Chain Rule

Logarithmic Properties as Aids to Differentiation

- Differentiate:

$f(x) = \ln\sqrt{x+1} = \ln(x+1)^{1/2}$

power rule

$= \frac{1}{2}\ln(x+1)$

$\frac{1}{2}\frac{d}{dx}\ln(x+1) =$

$\frac{1}{2}\left(\frac{1}{x+1}\right) = \boxed{\frac{1}{2(x+1)}}$

$f(x) = \ln\frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$. Log rules!

$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)$

$= \frac{1}{x} + 2\left(\frac{1}{x^2+1}\right)(2x) - \frac{1}{2}\left(\frac{1}{2x^3-1}\right)(6x^2)$

$= \boxed{\frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}}$

And yes, we can use logarithms to help us in differentiating nonlogarithmic functions!

Logarithmic Differentiation

Find the derivative of:

Take the log of the equation :)

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}} \quad , \ x \neq 2$$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1)$$

differentiate:

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right)\frac{(x^2+1)}{(x^2+1)} - \frac{1}{\cancel{2}}\left(\frac{\cancel{2}x}{x^2+1}\right)\left(\frac{x-2}{x-2}\right)$$

$$= \frac{2x^2+2-x^2+2x}{(x-2)(x^2+1)}$$

$$\frac{y'}{y} = \frac{x^2+2x+2}{(x-2)(x^2+1)}$$

$$y' = \frac{(x^2+2x+2)}{\cancel{(x-2)}(x^2+1)} \cdot \frac{(x-2)^{\cancel{2}}}{(x^2+1)^{1/2}}$$

$$\boxed{y' = \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}}}$$

Remember that logarithims are undefined for negative numbers, hence, you will find expressions with the $ln|x|$. That is OK, we are able to differentiate as if the absolute values signs were not present.

THEOREM 5.4 DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Derivative Involving Absolute Value

- Find the derivative of:

$$f(x) = \ln|\cos x|$$

$$= \frac{\frac{d}{dx}\cos x}{\cos x} = \frac{-\sin x}{\cos x} = \boxed{-\tan x}$$

Finding Relative Extrema

- Locate the Relative Extrema of:

$$y = \ln\left(x^2+2x+3\right)$$

$$y = \frac{2x+2}{x^2+2x+3} = 0 \qquad 2(x+1) = 0$$

$$x = -1$$

discriminant

$4 - 4(1)(3) =$ neg.

$-2 \quad -1 \quad 0$

$- \searrow \ \nearrow +$

$y = \ln(1-2+3) = \ln 2$

$\sim \ln 2$

$\boxed{\min\ (-1, \ln 2)}$