Calculus
Lesson 5.2: The Natural Logarithmic Function: Integration
Mrs. Snow, Instructor

$$\int \frac{1}{\text{cabin}}\, d\text{cabin} = \ln \text{cabin} + c$$

$$= \text{natural log cabin} + c$$

$$= \text{houseboat}$$

The differentiation rules that we studied in 5.1 produce the following integration rules.

THEOREM 5.5 LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

Using Log Rule for Integration

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$$

$$= 2\ln|x| + c$$

$$= \boxed{\ln(x^2) + c}$$

$$\int \frac{1}{4x-1}\,dx =$$

$u = 4x-1$

$du = 4\,dx$

$\frac{1}{4}du = dx$

$$\frac{1}{4}\int \frac{1}{u}\,du =$$

$$\frac{1}{4}\ln|u| + c =$$

$$\boxed{\frac{1}{4}\ln|4x-1| + c}$$

Finding Area with the Log Rule

- Find the area of the region bounded by the graph, the x-axis and the line x=3.

$y = \frac{x}{x^2+1} \rightarrow A = \int_0^3 \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int_0^3 \frac{1}{u}\,du =$

$\frac{1}{2}\ln|u| =$

$\frac{1}{2}\ln|x^2+1|\Big|_0^3 = \frac{1}{2}(\ln 10 - \cancel{\ln 1}^{0})$

$= \frac{1}{2}\ln 10 \sim 1.151$

Recognizing Quotient Forms of the Log Rule

a. $\int \frac{3x^2+1}{x^3+x}\,dx$

$u = x^3+x$

$du = (3x^2+1)dx$

$= \int \frac{du}{u} = \ln|u| + c = \boxed{\ln|x^3+x| + c}$

Remember both absolute values and +c

b. $\int \frac{\sec^2 x}{\tan x}\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\int \frac{1}{u}\,du = \ln|u| + c = \boxed{\ln|\tan x| + c}$

c. $\int \frac{x+1}{x^2+2x}\,dx =$

$u = x^2+2x$

$du = (2x+2)dx$

$\frac{1}{2}du = (x+1)dx$

$\frac{1}{2}\int \frac{1}{u}\,du =$

$\frac{1}{2}\ln\frac{1}{u} + c \Rightarrow \boxed{\frac{1}{2}\ln|x^2+2x| + c}$

d. $\int \frac{1}{3x+2}\,dx$

$u = 3x+2$

$du = 3dx$

$\frac{1}{3}du = dx$

$= \frac{1}{3}\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln u + c = \boxed{\frac{1}{3}\ln|3x+2| + c}$

If a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule.

Using Long Division Before Integrating

$$\int \frac{x^2+x+1}{x^2+1}dx =$$

$$x^2+1 \overset{\displaystyle 1}{\overline{\big)\, x^2+x+1}}$$
$$-(x^2 \quad +1)$$
$$x$$
$$= 1 + \frac{x}{x^2+1}$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = dx$

$$\int \left(1 + \frac{x}{x^2+1}\right)dx$$

$$\int dx + \int \frac{x}{x^2+1}dx$$

$$\int dx + \frac{1}{2}\int \frac{1}{u}du =$$

$$x + \frac{1}{2}\ln|u| + c = \boxed{x + \frac{1}{2}\ln|x^2+1| + c}$$

Change of Variables with the Log Rule

$$\int \frac{2x}{(x+1)^2}dx = 2\int \frac{x}{(x+1)^2}dx$$

$u = x+1 \rightarrow u-1 = x$

$du = dx$

$$= 2\int \frac{u-1}{u^2}du =$$

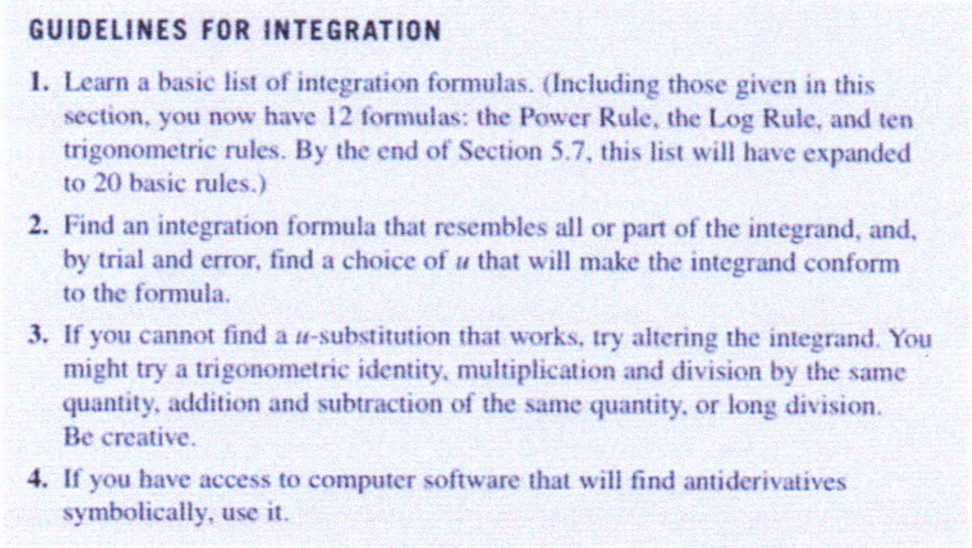
split into 2 fractions

$$= 2\int \frac{u}{u^2} - \frac{1}{u^2}du =$$

$$= 2\int \frac{1}{u}du - \int u^{\overset{+1}{-2}}du$$

$$= 2\left(\ln|u| + \frac{1}{u}\right) + C$$

$$= \boxed{2\ln|x+1| + \frac{2}{x+1} + C}$$



GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of *u* that will make the integrand conform to the formula.
3. If you cannot find a *u*-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

u-Substitution and the Log Rule

- Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

$$y = \int \frac{1}{x \ln x}\,dx \quad \text{rewrite}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int \frac{1/x}{\ln x}\,dx$$

$$= \int \frac{1}{u}\,du = \ln|u| + c = \boxed{\ln|\ln x| + c}$$

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

However .....

Using a Trigonometric Identity

$u = \cos x$

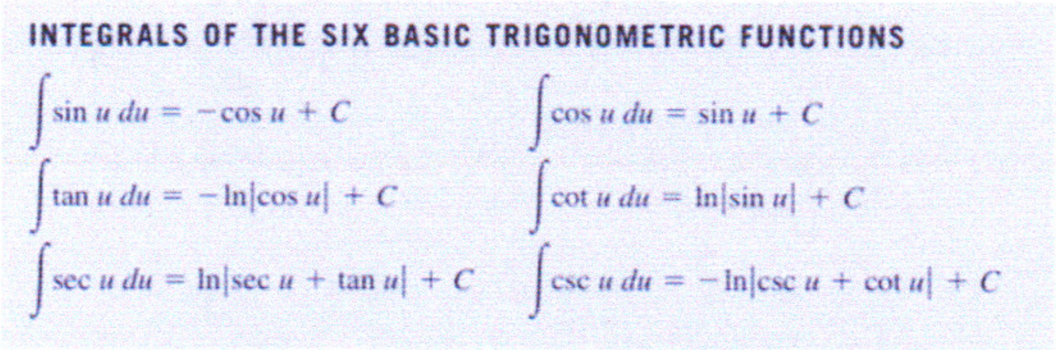
$du = -\sin x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + c$$

$$\boxed{= -\ln|\cos x| + c}$$



INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Integrating Trigonometric Functions

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$$

$$= \int_0^{\pi/4} \sec x\,dx$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sec x = \frac{\cancel{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \quad \sqrt{2}$$

$$= Ln\left|\sec x + \tan x\right| \Big|_0^{\pi/4}$$

$$= Ln\left|\left(\sec \pi/4 + \tan \pi/4\right) - Ln\left(\overset{1}{\cancel{\sec 0}} + \overset{0}{\cancel{\tan 0}}\right)\right.$$

$$= Ln\left(\sqrt{2}+1\right) \qquad -Ln\,1 \rightarrow 0$$

$$\approx .881$$

Finding Average Value

- Find the average value of $f(x) = tanx$ on the interval $\left[0, \frac{\pi}{4}\right]$ (Sec 4.4)

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \text{Avg Value}$$

$$\frac{1}{\pi/4 - 0}\int_0^{\pi/4} \tan x\,dx$$

$$= \frac{4}{\pi}\left(-\ln|\cos x|\right]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left(-Ln\cos\frac{\pi}{4} - (-Ln\cos 0\right)$$

$$\frac{4}{\pi}\left(-Ln\frac{\sqrt{2}}{2} + \overset{0}{\cancel{Ln\,1}}\right) = \frac{4}{\pi}\left(-Ln\frac{\sqrt{2}}{2}\right) \approx \boxed{.441}$$