Calculus
Lesson 4.4: The Fundamental Theorem of Calculus
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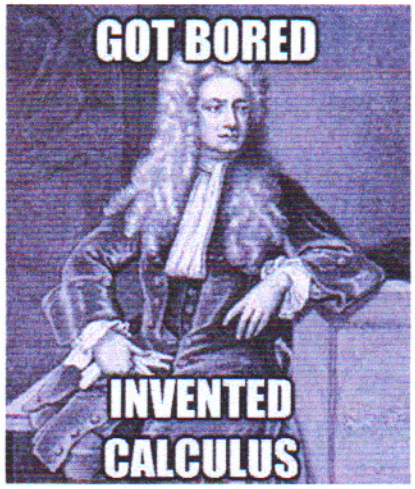
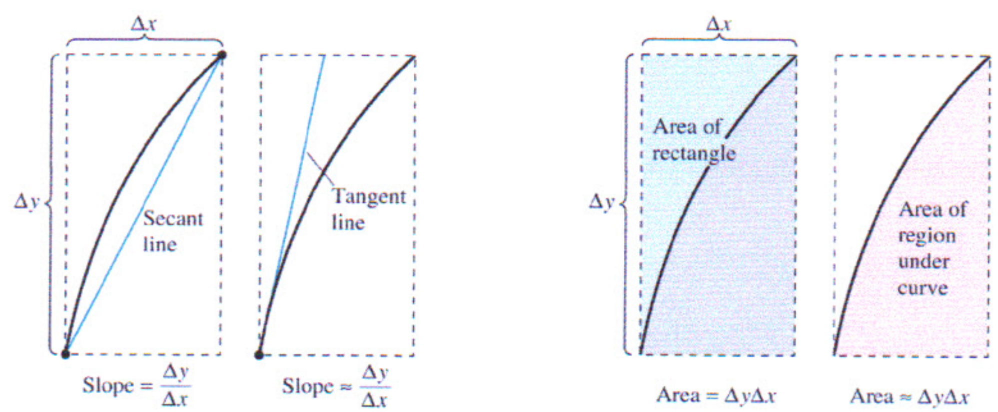


The two major branches of calculus are: differential calculus and integral calculus. At this point, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in a theorem that is appropriately called the **Fundamental Theorem of Calculus.** Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in the following figures. The slope of the tangent line was defined using the quotient $\frac{\Delta y}{\Delta x}$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the product $\Delta x \Delta y$ (the area of a rectangle).



(**a**) Differentiation (**b**) Definite integration

Differentiation and definite integration have an "inverse" relationship.

The limit process used to define the derivative and the definite integral preserve this inverse relationship.

THEOREM 4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b$$
$$= [F(b) + C] - [F(a) + C]$$
$$= F(b) - F(a).$$

Evaluate each definite integral.

$$\int_1^3 (x^2 - 3)\,dx = \frac{x^3}{3} - 3x\Big|_1^3 = \cancel{\frac{27}{3}} - \cancel{9} - \left(\frac{1}{3} - 3\right)$$
$$= -\frac{1}{3} + 3 = -\frac{1}{3} + \frac{9}{3} = \underline{\underline{\frac{8}{3}}}$$

$$\int_1^4 3\sqrt{x}\,dx = \int_1^4 3x^{\frac{1}{2}}dx = \frac{3x^{3/2}}{3/2} = \frac{2\cancel{3}x^{3/2}}{\cancel{3}} = 2x^{3/2}\Big|_1^4 =$$
$$= 16 - 2 = \underline{\underline{14}}$$

$$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0$$
$$= 1 - 0 = \underline{\underline{1}}$$

A definite integral involving absolute value

$$\int_0^2 |2x-1|\,dx =$$

$$|2x-1| = \begin{cases} -(2x-1) & x < \frac{1}{2} \\ (2x-1) & x \geq \frac{1}{2} \end{cases}$$

$$\int_0^{1/2} -(2x-1)\,dx + \int_{1/2}^{2} 2x-1\,dx$$

$$-x^2 + x \Big|_0^{1/2} + x^2 - x \Big|_{1/2}^{2} = -\frac{1}{4} + \frac{1}{2} + 4 - 2 - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\cancel{\frac{1}{4}} + \cancel{\frac{1}{2}} + 4 - 2 - \cancel{\frac{1}{4}} + \frac{1}{2} = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$$

Using the fundamental theorem to find area

Find the area of the region bounded by the graph $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines x=0 and x=2 $y > 0$ on interval $[0, 2]$

$$\int_0^2 2x^2 - 3x + 2\,dx = \frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x \Big|_0^2$$

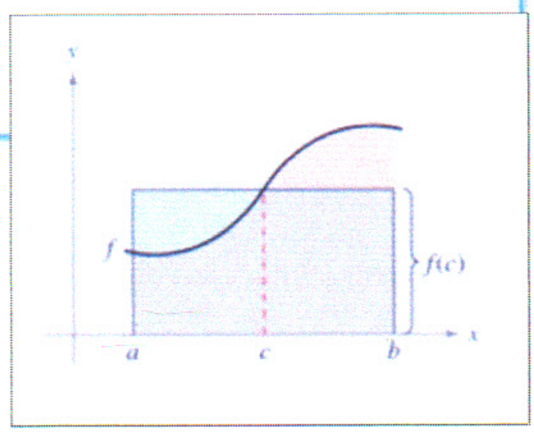
$$\frac{16}{3} - \frac{3}{\cancel{2}}(\cancel{4}^{\,2}) + 4 - 0 = \frac{16}{3} - 6 + 4$$

$$= \frac{16}{3} - \frac{18}{3} + \frac{12}{3} = \boxed{\frac{10}{3}}$$

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$



Using Mean Value Theorem

Find the values of c guaranteed by the mean value theorem for integrals over the given interval. $f(x) = \frac{9}{x^3}$, $[1, 3]$ $\quad \frac{9}{x^3} = 9x^{-3}$

$$\int_1^3 9x^{-3}\,dx = f(c)(3-1)$$

$$-\frac{9x^{-2}}{2}\Big|_1^3 = f(c)\,(2)$$

$$\frac{-\cancel{9}}{2(\cancel{9})} - \left(\frac{-9}{2(1)}\right) = \frac{-1}{2} + \frac{9}{2} = \frac{8}{2} = 4$$

$$4 = 2f(c)$$

$$\frac{4}{2} = 2 = f(c)$$

$$2 = \frac{9}{c^3}$$

$$\sqrt[3]{c^3} = \sqrt[3]{\frac{9}{2}} \Rightarrow c = 1.65$$

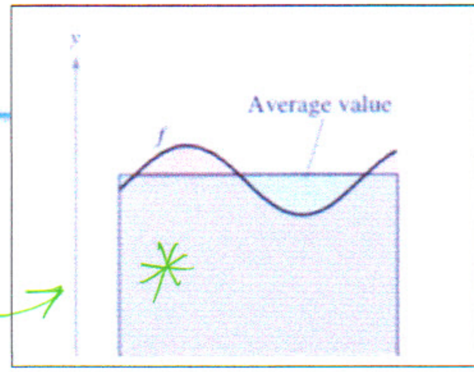
The value of $f(c)$given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Area under curve is * equal to the rectangle whose height is the average value



Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1,4]$.

$$\frac{1}{4-1}\int_1^4 (3x^2-2x)\,dx =$$

$$\frac{1}{3}\left[(x^3-x^2)\Big|_1^4\right] =$$

$$\frac{1}{3}\left[(64-16)-\cancel{(1-1)}\right] =$$

$$\frac{1}{3}(48) = \boxed{16}$$

The Definite Integral as a Function

Evaluate the function

$$F(x) = \int_0^x \cos t \; dt \quad = \sin t \Big|_0^x = \sin x - \sin 0 = \boxed{\sin x}$$

at

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \text{and } \frac{\pi}{2}$$

$\sin(0) = 0$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ & $\sin\left(\frac{\pi}{2}\right) = 1$

THEOREM 4.11 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Remember = Inverses undo each other = $\frac{d}{dx}$ undoes $\int$

← upper bound x

← lower bound is a constant

Using the Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}$$

Find the derivative of NOT x

$$\frac{d}{dx}\int_{\pi/2}^{x^3} \cos t\;dt$$

← just solve, one step at a time

Step 1 Integrate:

$$\int_{\frac{\pi}{2}}^{x^3} \cos t\,dt = \sin t\Big|_{\frac{\pi}{2}}^{x^3} = \sin x^3 - \sin\frac{\pi}{2}$$

$$= \sin x^3 - 1$$

Step 2 take derivative:

$$\frac{d}{dx}\sin x^3 - 1 = \left(\frac{d}{dx}x^3\right)\cos(x^3)$$ & chain rule!

$$= 3x^2\cos x^3$$

So ⇒ when upper bound other than x:

① Replace with upper bound

& ② take derivative of upper bound.

Why does this work?

Let $F(x) = \int_{\frac{\pi}{2}}^{x} \cos t\,dt$ (differentiate)

$F'(x) = \cos x$

$F'(x^3) = \cos x^3$

substitute

Our upper bound is x^3 so we have x^3

$$F(x^3) = \int_{\pi/2}^{x^3} \cos t\,dt$$

$$\frac{d}{dx}F(x^3) = F'(x^3)\,3x^2$$ chain rule

← what is $F'(x^3) = ?$

$$\therefore F'(x^3)\,3x^2 = \boxed{3x^2\cos x^3}$$