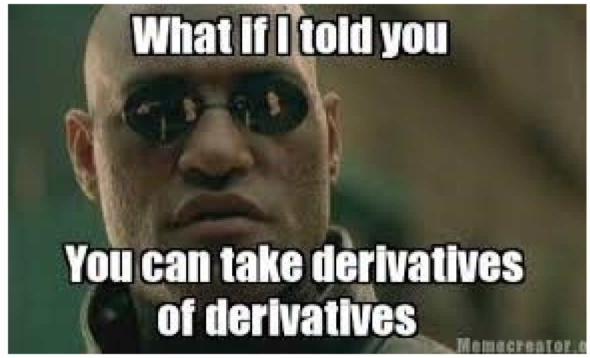
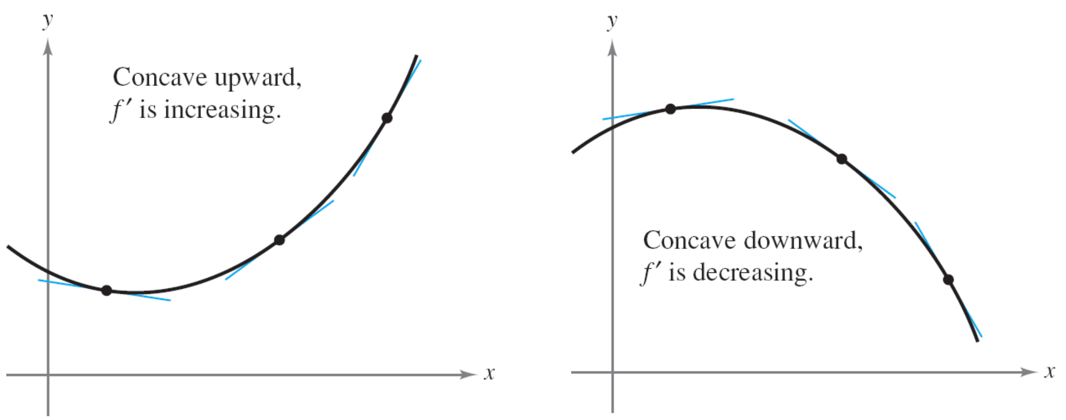
Calculus
Lesson 3.4: Concavity and the Second Derivative Test
Mrs. Snow, Instructor



In this section, we will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is curving upward or curving downward.

DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.



(a) The graph of f lies above its tangent lines.

(b) The graph of f lies below its tangent lines.

THEOREM 3.7 TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

To apply this theorem:

1. locate the x-vaues at which
   f"(x)=0 or $f\ "(x)\ does\ not\ exist\ (where\ the\ denominator\ is = 0$
2. use these these x-values to determine test intervals
3. test the sign of $f"(x)$ in each of the test intervals

Determining Concavity

Determine the open intervals on which the graph is concave up or concave down.

$$f(x) = \frac{6}{x^2 + 3}$$

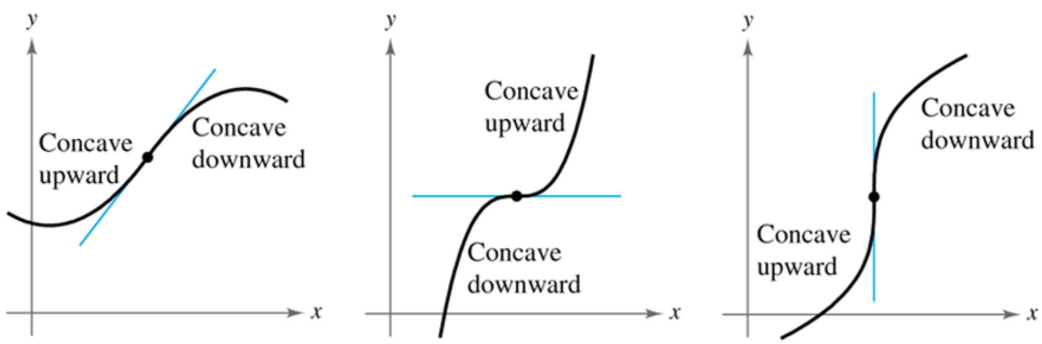
Determine the open intervals on which the graph is concave up or concave down.

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$

DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

The point where the concavity changes and the tangent line to the graph exists, is a **point of inflection.**



The concavity of f changes at a point of inflection. Note that a graph crosses its tangent line at a point of inflection.

To locate possible points of inflection, you can determine the values of x where $f''(x)=0$ or $f''(x)$ *does not exist*. The process is similar to locating extrema of f.

THEOREM 3.8 POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.

Note, the converse of this theorem is not necessarily true! Think of the parent quadratic function. $y = x^2$ is concave upwards from $-\infty < x < 0$ *and* $0 < x < \infty$, however its second derivative is 0 at $x = 0$ but, (0,0) is not a point of inflection.

Determine the points of inflection and discuss the concavity of the graph.

$f(x) = x^4 - 4x^3$

The seond derivative test may beused to preform a simple test for relative maxima and minima along with testing for concavity.

THEOREM 3.9 SECOND DERIVATIVE TEST

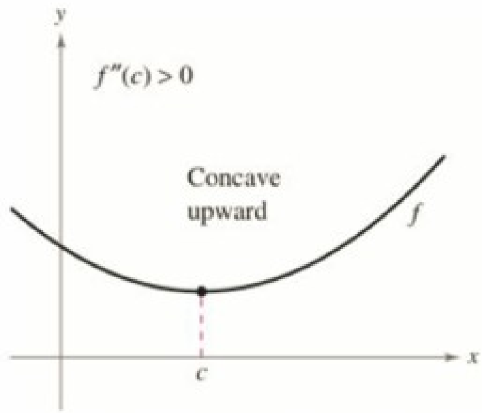
Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.

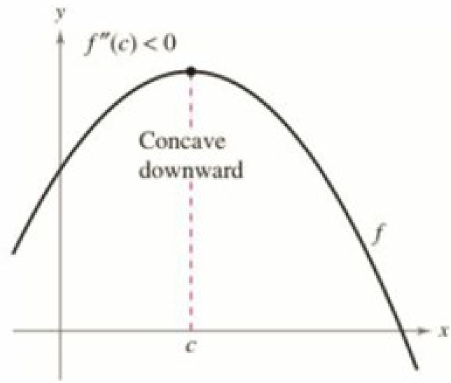
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

- Locate critical numbers These are where we may have a minimum or maximum or neither
- If f'' is positive, the function is concave up and c is a minimum
- If f'' is negative, the function is concave down and c is a maximum.
- If $f''=0$, c is neither minimum or maximum



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Find the relative extrema.

$$f(x) = -3x^5 + 5x^3$$