Calculus
Lesson 3.1: Extrema on an Interval, and
Lesson 3.2: Rolle's Theorem and the Mean Value Theorem
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Because calculus is used in so many applications of physics and engineering, finance, and so on, there is a lot of effort devoted to studying the behavior of a function over an interval. Where is the minimum value? Is there a minimum value? Where is the function increasing or decreasing? Is the object starting to slow down? These are just a few of the questions that may be answered by understanding what the extrema are on a given interval.

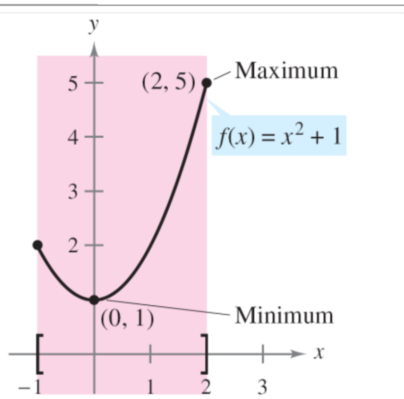
Definition of Extrema

DEFINITION OF EXTREMA

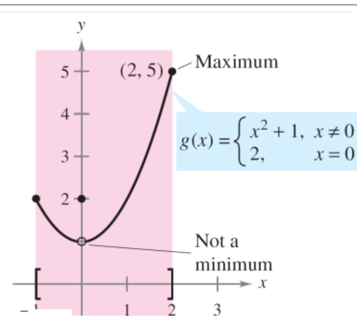
Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \ge f(x)$ for all x in I.

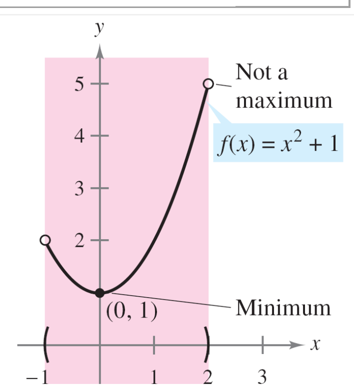
The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval.



(a) f is continuous, $[-1, 2]$ is closed.



(c) g is not continuous, $[-1, 2]$ is closed.
Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**



(b) f is continuous, $(-1, 2)$ is open.

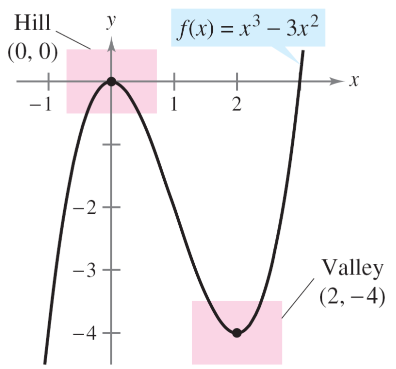
THEOREM 3.1 THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

DEFINITION OF RELATIVE EXTREMA

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum,** respectively.

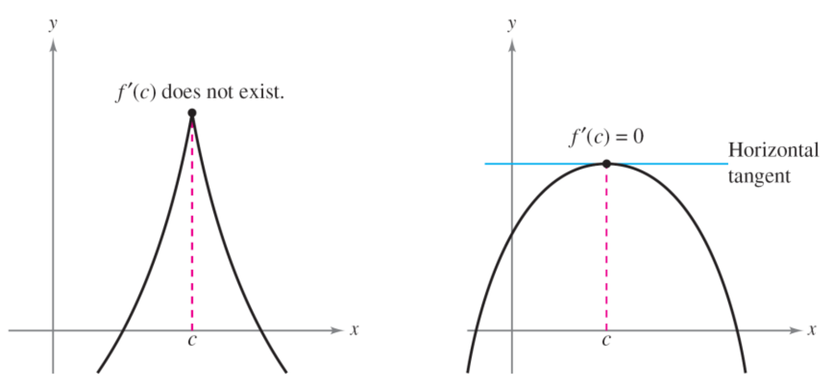


f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.

Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph.

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.



c is a critical number of f.

THEOREM 3.2 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

These "hills" and "valleys" can occur in two ways. If the function is smooth and rounded, the graph has a horizontal tangent line at the minimum or maximum. If it is sharp and peaked, the graph represents a function that is not differentiable and the minimum or maximum occurs at the peak.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

Finding Extrema on a Closed Interval

$f(x) = 3x^4 - 4x^3 \;\; on\ the\ interval\ [-1,2]$

- find $f'(x) = 0$
- evaluate $f(x)$ at the critical numbers within the interval and the endpoints

$f(x) = 2x - 3x^{2/3} \;\; on\ the\ interval\ of\ [-1,3]$

$f(x) = 2\sin x - \cos 2x \;\; on\ the\ interval\ [0, 2\pi]$

3.2: Rolle's Theorem and the Mean Value Theorem

The Extreme Value Theorem from 3.1, states that on a closed interval, there will be both a minimum and a maximum, and they may occur on an endpoint. **Rolle's Theorem,** named after the French mathematician Michel Rolle (1652–1719), gives specific conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

THEOREM 3.3 ROLLE'S THEOREM

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c) = 0$.

$f(x) = x^4 - 2x^2$ Find all values of c in the interval $(-2,2)$ such that $f'(c) = 0$

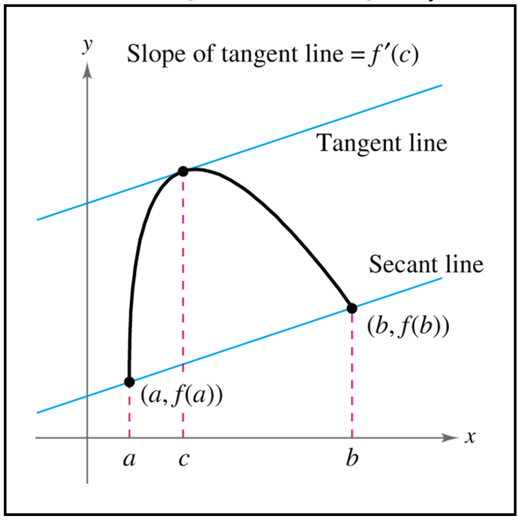
The Mean Value Theorem

THEOREM 3.4 THE MEAN VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The "mean" refers to the mean or average rate of change of f in the interval $[a, b]$



The Mean Value Theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))and\ (b, f(b))$ For rate of change, ther must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the close interval $[a, b]$

Finding a Tangent Line

$f(x) = 5 - 4/x$ find all values of c in the open interval $(1,4)$ such that:

$f'(c) = \frac{f(4)-f(1)}{4-1}$

in other words, to find the tangent line we will work backwards to find the value of $f'$and with the slope determine the value of x.