Calculus
Lesson 2.4 The Chain Rule
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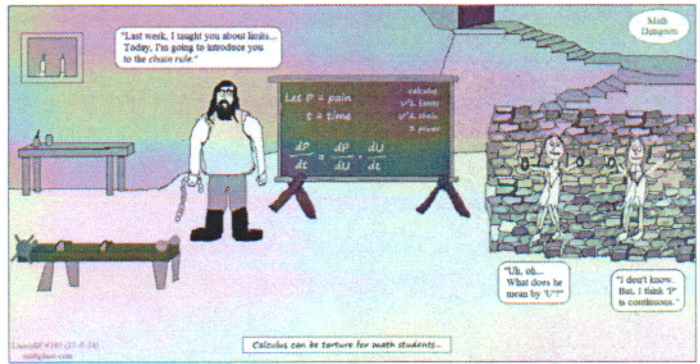

How do you take the derivative of: $F(x) = \sqrt{x^2 + 1}$???

Well, good question, maybe no answer? The differentiation formulas we have seen so far will not enable us to calculate $F'(x)$. F is a composite function. If we let the outside part be $y = f(u) = \sqrt{u}$ and the inside part be $u = g(x) = x^2 + 1$, we can then write:

$y = F(x) = f(g(x)) = f \circ g$. So!!! It would be nice to have a rule that tells us how to find the derivative of $F = f \circ g$. This is where the **chain rule** comes into play and it will find the derivative of F in terms of f and g! Fact is, the **chain rule** is one of the most important of the differentiation rules.

$y = f(u) = \sqrt{u}$

$\frac{du}{dy} = \frac{d}{du}\sqrt{u}$

THEOREM 2.10 THE CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\left(\frac{d}{dx} \text{ outside}\right)\left(\frac{d}{dx} \text{ inside}\right)$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x). = \left(f'(u)\right)\left(g'(x)\right)$$

$y = f(g(x))$	$u = g(x)$ (inside)	$y = f(u)$ Outside
a. $y = \frac{1}{x+1}$	$x+1$	$\frac{1}{x}$
b. $y = \sin 2x$	$2x$	$\sin x$
c. $y = \sqrt{3x^2 - x + 1}$	$3x^2 - x + 1$	$\sqrt{x}$
d. $y = \tan^2 x = (\tan x)^2$	$\tan x$	x^2

Special Case:
When the composite function involves an exponent, like in *example "d"*, $y = u^2$, we have a special case of the Chain Rule which uses the Power Rule:

THEOREM 2.11 THE GENERAL POWER RULE

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

Applying the General Power Rule:
$y = (x^2 + 1)^3$

inside function:
$u = x^2+1$
outside function:
$f(u) = x^3$

$$y' = \underbrace{3(x^2+1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = \underline{6x(x^2+1)^2}$$

Find the derivative:
$f(x) = (3x-2x^2)^3$

$$f'(x) = 3\underbrace{(3x-2x^2)^2}_{\frac{dy}{du}}\underbrace{(3-4x)}_{\frac{du}{dx}} = \boxed{(9-12x)(3x-2x^2)^2}$$

Differentiating Functions Involving Radicals
Find all points on the graph of $f(x)$ for which $f'(x) = 0$ ② and those for which $f'(x)$ does not exist. ③

$$f(x) = \sqrt[3]{(x^2-1)^2} = (x^2-1)^{2/3}$$

$$\frac{2}{3} - 1 = -\frac{1}{3}$$

$$f' = \frac{2}{3}\underbrace{(x^2-1)^{2/3-1}}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}}$$

① $$f'(x) = \frac{4x}{3(x^2-1)^{1/3}}$$

② $$\frac{4x}{3(x^2-1)^{1/3}} = 0$$
$4x = 0$
$x = 0$
$\boxed{(0, 1)}$
go back to $f(x)$

③ $3\sqrt[3]{x^2-1} = 0$ $f'(x)$ DNE at denom $= 0$
$x^2 - 1 = 0$
$x = \pm 1$ $\boxed{(\pm 1, 0)}$

Differentiating Quotients with Constant Numerators

$$g(t)=\frac{-7}{(2t-3)^2} = -7(2t-3)^{-2}$$

$$g' = (-2)(-7)(2t-3)^{-3}(2)$$

$$g'(t) = \frac{28}{(2t-3)^3}$$

Simplifying by Factoring Out the Least Powers Product & chain Rule

$$f(x)=x^2\sqrt{1-x^2} = x^2(1-x^2)^{1/2}$$

$$f' = 2x(1-x^2)^{1/2} + \tfrac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)(x^2)$$

"Least Power"

$$= 2x(1-x^2)^{1/2} - (1-x^2)^{-1/2}x^3$$

Factor out $(1-x^2)^{-1/2}$

Distribute

$$f' = (1-x^2)^{-1/2}\left[2x(1-x^2)^{1} - x^3\right]$$

so? what is left??

move below

$$= \frac{2x-2x^3-x^3}{(1-x^2)^{1/2}} = \frac{2x-3x^3}{(1-x^2)^{1/2}} = \boxed{\frac{x(2-3x^2)}{(1-x^2)^{1/2}}}$$

& simplify

Exponent Rules:

$$(1-x^2)^{-1/2}(1-x^2)^{\boxed{1}} = (1-x^2)^{1/2}$$

$$-\tfrac{1}{2} + \square = 1/2$$

$$\therefore\ \square = 1$$

Simplifying the Derivative of a Quotient

$$f(x)=\frac{x}{\sqrt[3]{x^2+4}} = \frac{x}{(x^2+4)^{1/3}}$$

$$f' = \frac{(1)(x^2+4)^{1/3} - \frac{1}{3}x(x^2+4)^{-2/3}(2x)}{\left((x^2+4)^{1/3}\right)^2} =$$

factor out: $\frac{1}{3}(x^2+4)^{-2/3}$

$$= \frac{(x^2+4)^{1/3} - \frac{1}{3}(x^2+4)^{-2/3}(2x^2)}{(x^2+4)^{2/3}}$$

$$= \frac{\frac{1}{3}(x^2+4)^{-2/3}\left[3(x^2+4)^{1} - 2x^2\right]}{(x^2+4)^{2/3}} =$$

expand & simplify

move below

$$= \frac{3x^2+12-2x^2}{3(x^2+4)^{2/3}(x^2+4)^{2/3}} = \boxed{\frac{x^2+12}{3(x^2+4)^{4/3}}}$$

Simplifying the Derivative of a Power

$$y=\left(\frac{3x-1}{x^2+3}\right)^2$$

$$y' = 2\left(\frac{3x-1}{x^2+3}\right)\frac{d}{dx}\left(\frac{3x-1}{x^2+3}\right)$$

$$= 2\left(\frac{3x-1}{x^2+3}\right)\left[\frac{3(x^2+3)-2x(3x-1)}{(x^2+3)^2}\right]$$

$$= 2\left(\frac{3x-1}{x^2+3}\right)\left[\frac{3x^2+9-6x^2+2x}{(x^2+3)^2}\right]$$

$$y' = \boxed{\frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}}$$

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u]=(\cos u)u' \qquad \frac{d}{dx}[\cos u]=-(\sin u)u'$$

$$\frac{d}{dx}[\tan u]=(\sec^2 u)u' \qquad \frac{d}{dx}[\cot u]=-(\csc^2 u)u'$$

$$\frac{d}{dx}[\sec u]=(\sec u\tan u)u' \qquad \frac{d}{dx}[\csc u]=-(\csc u\cot u)u'$$

Memorize

Applying the Chain Rule to Trigonometric Functions

a. $y=\sin 2x$ u

$y' = \cos 2x\left(\frac{d}{dx}2x\right)$

$= \underline{2\cos 2x}$

b. $y=\cos(x-1)$ u

$y' = -\sin(x-1)\left(\frac{d}{dx}(x-1)\right)$

$= \underline{-\sin x-1}$

c. $y=\tan 3x$ u

$y' = \sec^2 3x\left(\frac{d}{dx}3x\right)$

$y' = \underline{3\sec^2 3x}$

Parentheses and Trigonometric Functions coefficient

a. $y = \cos 3x^2$

$y' = (-\sin 3x^2)(6x)$

$= -6x \sin 3x^2$

b. $y = (\cos 3)x^2$

$y' = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos 9x^2$

$y' = (-\sin(9x^2))(18x)$

$= -18x \sin 9x^2$

d. $u = \cos^2 x = (\cos x)^2$

$u' = 2\cos x(-\sin x)$

$= -2\cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$

$y' = \frac{1}{2} \cos x^{-\frac{1}{2}}(-\sin x)$

$= \dfrac{-\sin x}{2(\cos x)^{1/2}}$

Repeated Application of the Chain Rule

$f(t) = \sin^3 4t = (\sin 4t)^3$

$f'(t) = 3(\sin 4t)^2(\cos 4t)(4)$

$= 12 \sin^2 4t \cos 4t$

Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of $f(x)$ at the point $(\pi, 1)$.
Then determine all values of x in the interval $(0,2\pi)$ at which the graph of f has a horizontal tangent.

$f(x) = 2\sin x + \cos 2x$

$f' = 2\cos x - 2\sin 2x = m$ at pt $(\pi, 1)$

$= 2\cos\pi - 2\cancel{\sin 2\pi}_0$

$= 2(-1) = -2 = m$

$y = mx + b$

$1 = -2(\pi) + b$

$1 + 2\pi = b$

$\boxed{y = -2x + 2\pi + 1}$

X for m=0
(horizontal tangent line)

$2\cos x - 2\sin 2x = 0$

$2\cos x - 2(2\sin x \cos x) = 0$

$2\cos x(1 - 2\sin x) = 0$

$2\cos x = 0$ $\quad$ $1 - 2\sin x = 0$

$\cos x = 0$ $\quad$ $\sin x = \frac{1}{2}$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2} \quad x = \frac{\pi}{6}, \frac{5\pi}{6}}$

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules	Let f, g, and u be differentiable functions of x.		
	Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$	*Sum or Difference Rule:* $\frac{d}{dx}[f \pm g] = f' \pm g'$	
	Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$	*Quotient Rule:* $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$	
Derivatives of Algebraic Functions	*Constant Rule:* $\frac{d}{dx}[c] = 0$	*(Simple) Power Rule:* $\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$	
Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\tan x] = \sec^2 x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$
	$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$
Chain Rule	*Chain Rule:* $\frac{d}{dx}[f(u)] = f'(u)\, u'$	*General Power Rule:* $\frac{d}{dx}[u^n] = nu^{n-1}\, u'$	