

You know how to find derivatives of polynomial functions. This packet will show you how to find the derivatives of exponential functions, then products and quotients of polynomials.

First, we need a few shortcuts for differentiation.

The Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Derivative of a^x

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

The Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

The Product Rule

The Product Rule

$$(fg)' = f'g + fg'$$

The derivative of a product is the derivative of the first factor multiplied by the second, plus the first factor multiplied by the derivative of the second.

Example 1. Differentiate $f(x) = x^2(x^3 + 5)$

$$\begin{aligned} \text{Solution: } f'(x) &= \frac{d}{dx}(x^2(x^3 + 5)) = \left[\frac{d}{dx}x^2 \cdot (x^3 + 5) \right] + \left[x^2 \cdot \frac{d}{dx}(x^3 + 5) \right] \\ &= 2x \cdot (x^3 + 5) + x^2 \cdot (3x^2) \\ &= 2x^4 + 10x + 3x^4 \\ &= 5x^4 + 10x \end{aligned}$$

Check: multiply it out and then differentiate $f(x) = x^2(x^3 + 5) = x^5 + 5x^2$

$$f'(x) = \frac{d}{dx}(x^5 + 5x^2) = 5x^4 + 10x$$

Example 2. Differentiate $y = x \cdot 2^x$

$$\begin{aligned} \text{Solution: } y' &= \frac{d}{dx}(x \cdot 2^x) = \frac{d}{dx}x \cdot 2^x + x \cdot \frac{d}{dx}2^x \\ &= 1 \cdot 2^x + x \ln 2(2^x) \\ &= 2^x + x \ln 2(2^x) \quad \text{<Factor out } 2^x \text{ from each term>} \\ &= (1 + x \ln 2)2^x \end{aligned}$$

Example 3. Differentiate $y = (t^2 + 3)e^t$

$$\begin{aligned} \text{Solution: } y' &= \frac{d}{dt}(t^2 + 3)e^t = \frac{d}{dt}(t^2 + 3) \cdot e^t + (t^2 + 3) \frac{d}{dt}(e^t) \\ &= (2t)e^t + (t^2 + 3)e^t \quad \text{<Factor out } e^t \text{ from each term>} \\ &= (2t + t^2 + 3)e^t \\ &= (t^2 + 2t + 3)e^t \end{aligned}$$

The Quotient Rule

The Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

The derivative of a quotient is the derivative of the numerator times the denominator minus the numerator time the derivative of the denominator all over the denominator squared.

Example 4. Differentiate $f(x) = \frac{1+x}{2+3x+4x^2}$

$$\begin{aligned} \text{Solution: } f'(x) &= \frac{d}{dx} \left(\frac{1+x}{2+3x+4x^2} \right) = \frac{\left(\frac{d}{dx}(1+x)\right) \cdot (2+3x+4x^2) - (1+x) \cdot \left(\frac{d}{dx}(2+3x+4x^2)\right)}{(2+3x+4x^2)^2} \\ &= \frac{1 \cdot (2+3x+4x^2) - (1+x)(3+8x)}{(2+3x+4x^2)^2} \\ &= \frac{(2+3x+4x^2) - (3+11x+8x^2)}{(2+3x+4x^2)^2} \\ &= \frac{-4x^2 - 8x - 1}{(2+3x+4x^2)^2} \end{aligned}$$

Example 5. Differentiate $y = \frac{3x+1}{5x+2}$

$$\begin{aligned} \text{Solution: } y' &= \frac{d}{dx} \left(\frac{3x+1}{5x+2} \right) = \frac{\left(\frac{d}{dx}(3x+1)\right) \cdot (5x+2) - (3x+1) \left(\frac{d}{dx}(5x+2)\right)}{(5x+2)^2} \\ &= \frac{3(5x+2) - (3x+1) \cdot 5}{(5x+2)^2} \\ &= \frac{(15x+6) - (15x+5)}{(5x+2)^2} \\ &= \frac{15x+6-15x-5}{(5x+2)^2} \\ &= \frac{1}{(5x+2)^2} \end{aligned}$$

Example 6. Differentiate $f(x) = \frac{1}{1+e^x}$

$$\begin{aligned} \text{Solution: } f'(x) &= \frac{d}{dx} \left(\frac{1}{1+e^x} \right) = \frac{\left(\frac{d}{dx}(1)\right) \cdot (1+e^x) - (1) \left(\frac{d}{dx}(1+e^x)\right)}{(1+e^x)^2} \\ &= \frac{0(1+e^x) - 1(0+e^x)}{(1+e^x)^2} \\ &= \frac{-e^x}{(1+e^x)^2} \end{aligned}$$

Product and Quotient Rules

Section 1. For Problems 1-10, find the derivative **using the product rule**. (5 points each)

1. $f(x) = (3x + 8)(2x - 5)$

1. _____

2. $f(x) = xe^x$

2. _____

3. $y = (x^2 + 7) \cdot 2^x$

3. _____

4. $y = (t^3 - 7t^2 + 1)e^t$

4. _____

5. $y = (x^2 - 1)(x^2 + 7x - 8)$

5. _____

6. $w = (t^3 + 5t)(t^2 - 7t + 2)$

6. _____

7. $f(t) = 2^t(1 - t)$

7. _____

8. $y = e^x(3 - x^2)$

8. _____

9. $f(x) = (3x^2 + 5x)e^x$

9. _____

10. $f(x) = \frac{e^x}{x^2}$ (Hint: rewrite with a negative exponent)

10. _____

Section 2. For Problems 11-18, find the derivative **using the quotient rule.** (5 points each)

11. $f(x) = \frac{5x^2}{x^3 + 1}$

11. _____

12. $y = \frac{x}{e^x}$

12. _____

13. $z = \frac{t^2 + 5t + 2}{t + 3}$

13. _____

14. $f(x) = \frac{x^2 + 3}{x}$

14. _____

15. $f(z) = \frac{3z^2}{5z^2 + 7z}$

15. _____

16. $f(x) = \frac{1 + x}{x + 3}$

16. _____

17. $y = \frac{5x}{x^2 + 7}$

17. _____

18. $f(x) = \frac{e^x}{x^2}$

18. _____

Section 3. If $f(x) = 2^x \cdot 3^x$, find $f'(x)$ and $f''(x)$ (5 points each)

19. $f'(x) =$

19. _____

20. $f''(x) =$

20. _____