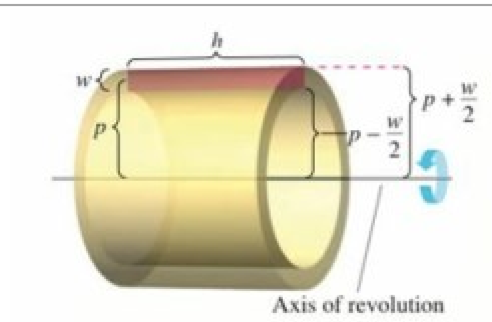
**Calculus**
**Lesson 7.3: Volume: The Shell Method**
**Mrs. Snow, Instructor**


In this section, you will study an alternative method for finding the volume of a solid of revolution. This method is called the shell method because it uses cylindrical shells. There are advantages and disadvantages between the shell and disk methods. More to come on this topic.




When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness= $w$.

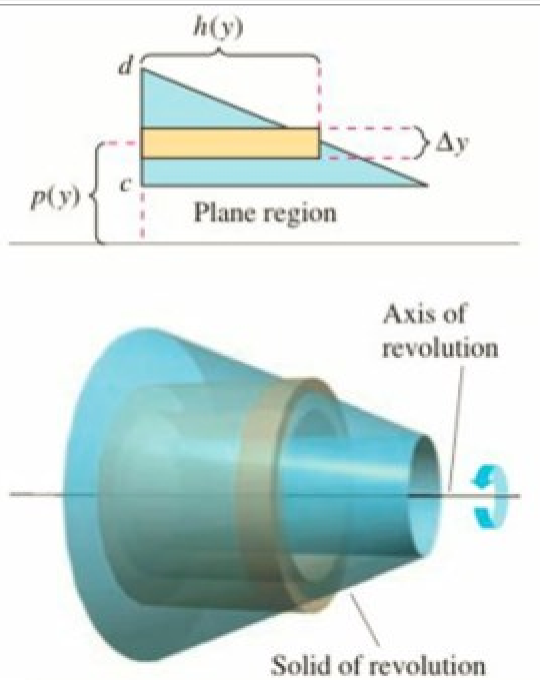
So from geometry we have the formula for the area of a cylinder:

$$A = 2\pi rh$$

With these shells/cylinders we have both the surface area AND a thickness of $\Delta \boldsymbol{x}$ ***or*** $\Delta \boldsymbol{y}$. Volume of the shells or cylinders takes on the form of:

$$\Delta \boldsymbol{V} = \boldsymbol{2\pi}[\boldsymbol{r}(\boldsymbol{y})\boldsymbol{h}(\boldsymbol{y})]\Delta \boldsymbol{y}$$

$$V = 2\pi(average\ radius)(height)(thickness)$$



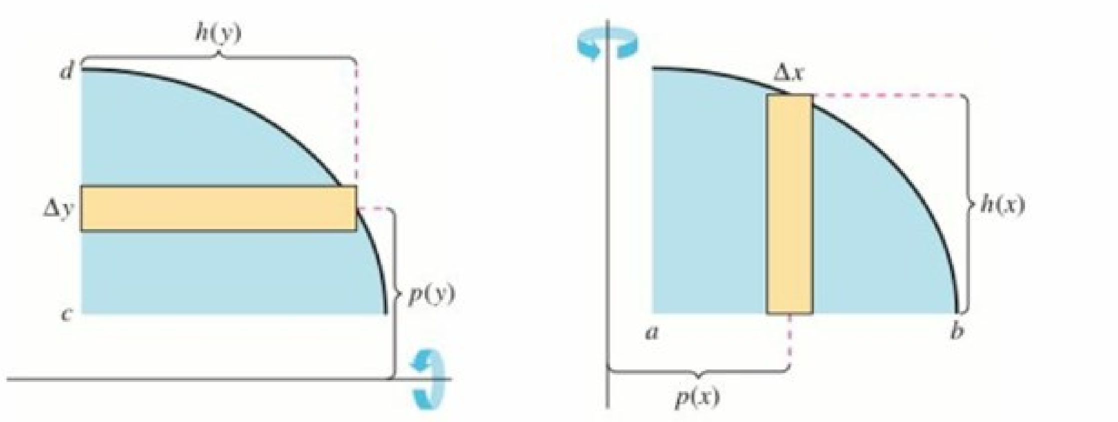

You can use this formula to find the volume of a solid of revolution. Assume that the plane region in the figure to the left is revolved about a line to form the indicated solid. If you consider a horizontal rectangle of width $\Delta y$, then, as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is:

$$\Delta V = 2\pi[r(y)h(y)]\Delta y$$

## THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 7.29.

| *Horizontal Axis of Revolution* | *Vertical Axis of Revolution* |
|---|---|
| Volume $= V = 2\pi \int_c^d p(y)h(y)\,dy$ | Volume $= V = 2\pi \int_a^b p(x)h(x)\,dx$ |
| $V = 2\pi \int_c^d r(y)h(y)dy$ | $V = 2\pi \int_a^b r(x)h(x)dx$ |




Horizontal axis of revolution

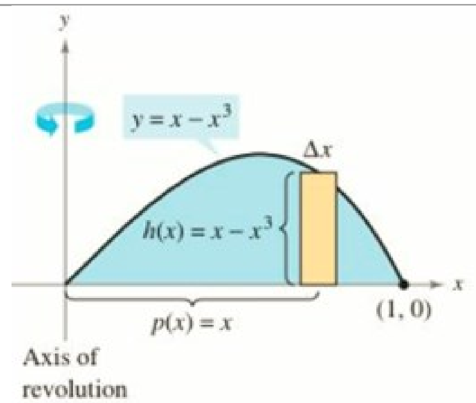
Vertical axis of revolution

**Figure 7.29**

**Using the Shell Method to find Volume**

- Find the volume of the solid of revolution formed by revolving the region bounded by y and the x-axis about the y-axis.

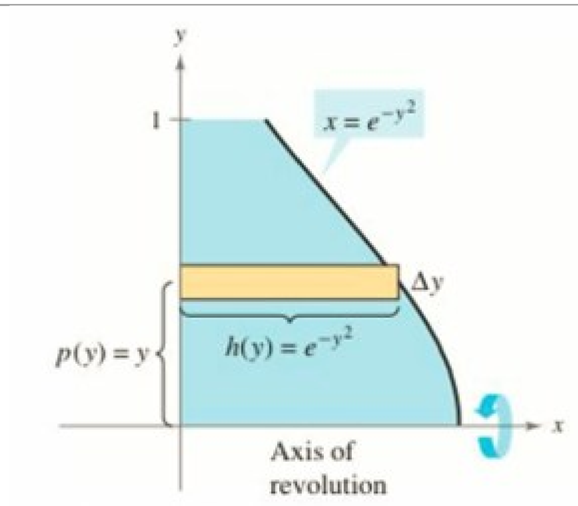
$y = x - x^3$ $0 \le x \le 1$



**Using the Shell Method to Find Volume**

- Find the volume of the solid of revolution formed by revolving the region bounded by the graph of x and the y-axis about the x-axis.

$x = e^{-y^2}$ $\quad\quad 0 \le y \le 1$

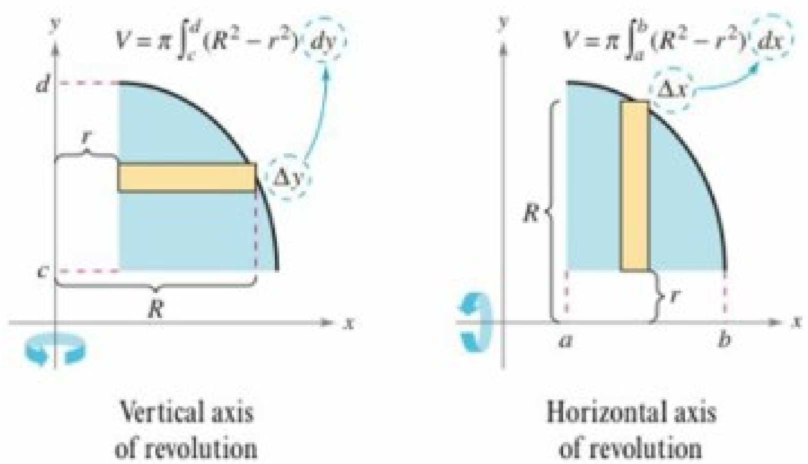


**Comparison of the Disk and Shell Methods**

The disk and shell methods can be distinguished as follows.

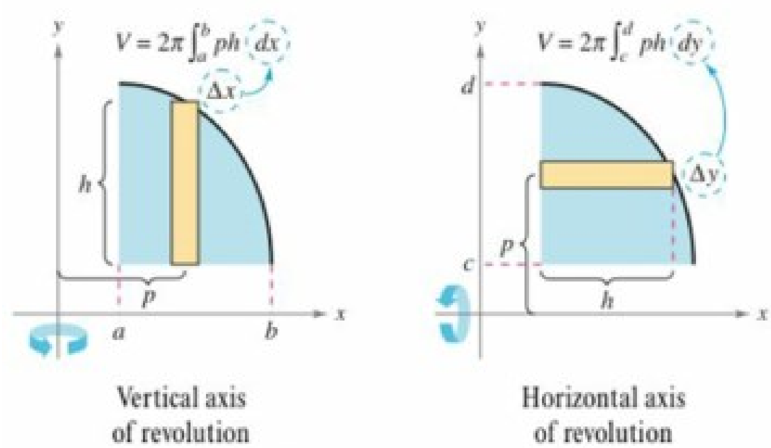
- For the disk method, the representative rectangle is always ***perpendicular*** to the axis of revolution.
- For the shell method, the representative rectangle is always ***parallel*** to the axis of revolution, as shown in the figures below.

**Disk Method:**




Disk method: Representative rectangle is perpendicular to the axis of revolution.

**Shell Method:**



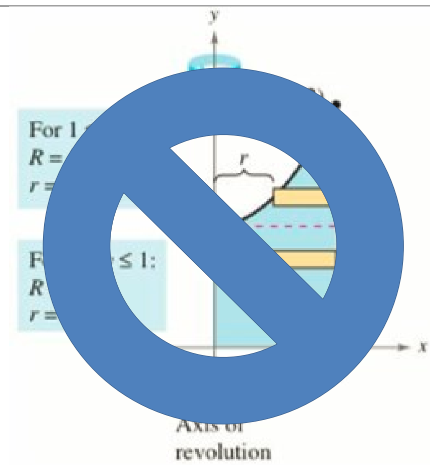

Shell method: Representative rectangle is parallel to the axis of revolution.

Often, one method is more convenient to use than the other. The following example illustrates a case in which the shell method is preferable. Here using the disk method, we would need two integrals to find the volume.

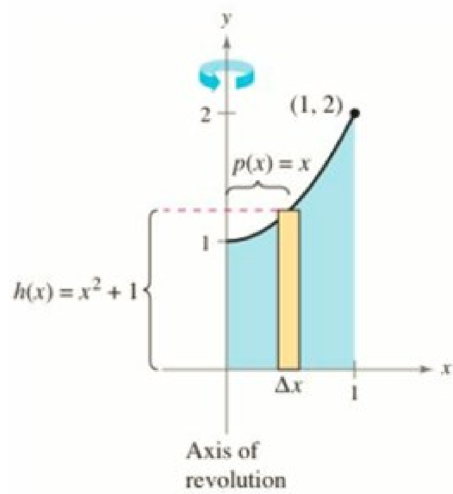
**Shell Method Preferable**

- Find the volume of the solid formed by revolving the region bounded by the following graphs about the y-axis.

$$y = x^2 + 1, \qquad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$




**(a)** Disk method



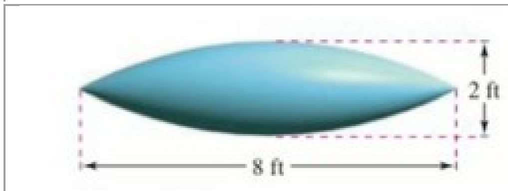

**(b)** Shell method

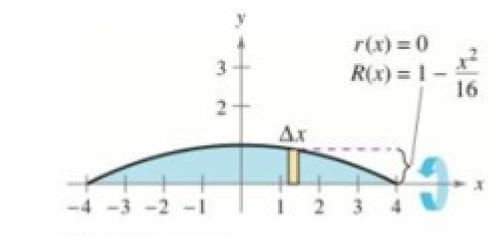
Here the **Disk Method is preferable** using only one integral compared the Shell Method which would require two integrals.

**Volume of a Pontoon: Disk Method is preferable**

- A pontoon is designed by rotating the graph about the x-axis, where x and y are measured in feet. Find the volume of the pontoon.

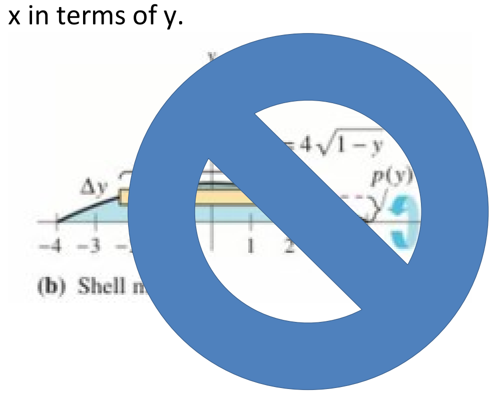
$$y = 1 - \frac{x^2}{16}, \qquad -4 \le x \le 4$$



(a) Disk method

Here you will have to solve the equation for x in terms of y.



Sometimes, solving for x is very difficult (or even impossible). In such cases you must use a vertical rectangle (of width $\Delta x$), thus making $x$ the variable of integration. The position (horizontal or vertical) of the axis of revolution then determines the method to be used.

**Shell Method Necessary**

Find the volume of the solid formed by revolving the region bounded by the graphs about the line x=2

$$y = x^3 + x + 1, \quad y = 1, \quad \text{and} \quad x = 1$$