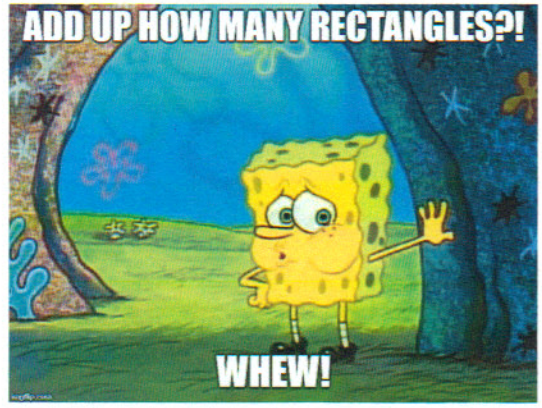
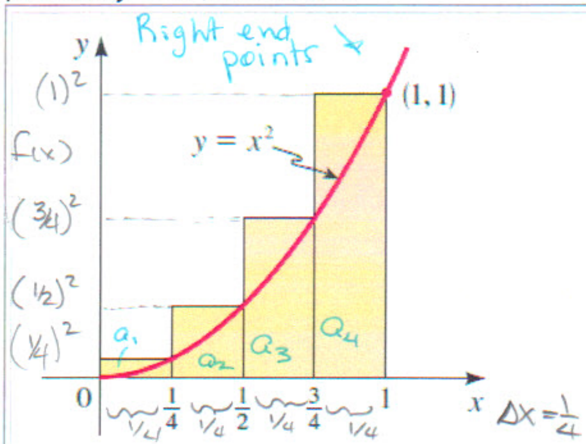


Precalculus
Lesson 14.5: The Area Problem: The Integral
Mrs. Snow, Instructor

In geometry we found area of polygons. We had set formulas such as the area of a rectangle is length times width. A triangular area is found by calculating ½ the length of the base times the height, and so on. Calculus is used to deal with area problems that have regions containing curved boundaries. Here we can go back to our simple formula for the area of a rectangle and use it to estimate the area of a region under a curve.

Estimating an Area Using Rectangles: Use rectangles to estimate the area under the parabola $y = x^2$, from 0 to 1.



Area = (width)(height)

$a_1 = \frac{1}{4}\left(\frac{1}{4}\right)^2$
$a_2 = \frac{1}{4}\left(\frac{1}{2}\right)^2$
$a_3 = \frac{1}{4}\left(\frac{3}{4}\right)^2$
$+\, a_4 = \frac{1}{4}(1)^2$

Total area
Add

$A =$
$= \frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{9}{16}\right) + \frac{1}{4}(1)$

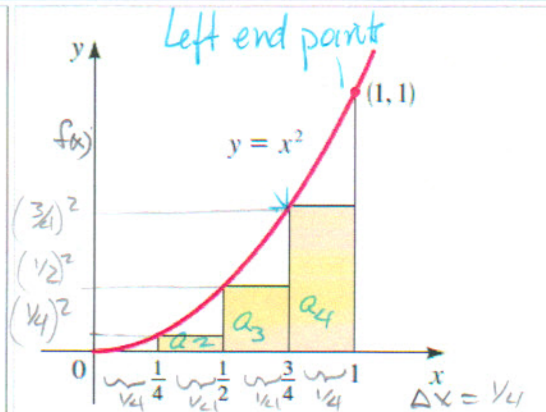
common factor - pullout

$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right)$

$= \frac{1}{4}\left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16}\right) = \frac{1}{4}\left(\frac{30}{16}\right)$

$R_4 = 46875^2$ units^2

R = right end points "4" = 4 rectangles
NOTE: OVER ESTIMATE.



$a_1 = 0$

Note: only 3 rectangles

$a_2 = \frac{1}{4}\left(\frac{1}{4}\right)^2$
$a_3 = \frac{1}{4}\left(\frac{1}{2}\right)^2$
$+\, a_4 = \frac{1}{4}\left(\frac{3}{4}\right)^2$

ADD

$A = \frac{1}{4}\left(\frac{1}{4}^2 + \frac{1}{2}^2 + \frac{3}{4}^2\right)$

$= \frac{1}{4}\left(\frac{1}{16} + \cancel{\frac{1}{4}} + \frac{9}{16} + \frac{4}{16}\right)$

$= \frac{1}{4}\left(\frac{14}{16}\right) =$

$L_4 = .21875$ units^2

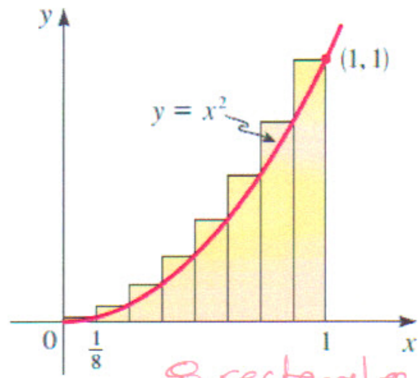
L = left endpoints, divided into 4 rectangles

Area is approximate!

$\boxed{.21875 < A < .46875}$

Same problem....smaller rectangles $\Rightarrow$ more rectangles $\therefore$ a more accurate estimate

Overestimate of area.

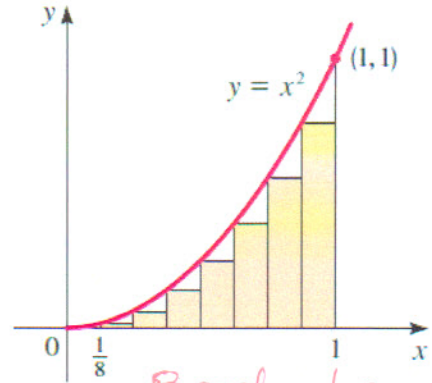


8 rectangles

(b) Using right endpoints

R_8

Underestimate of area.

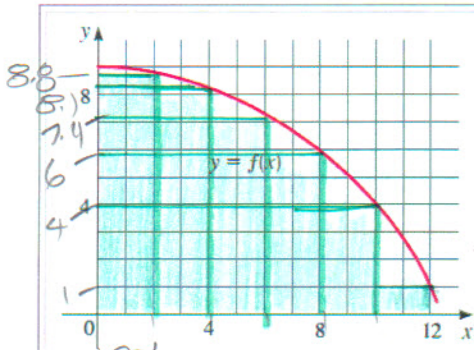


8 rectangles

(a) Using left endpoints

L_8

The smaller the rectangular strips the more accurate the calculation of the area. This then opens up the door to take a limit as the number of rectangles goes to infinity.



$\Delta x = 2 = \frac{12-0}{6} = 2$

$a_1 = 2(8.8)$
$a_2 = 2(8.1)$
$a_3 = 2(7.4)$
$a_4 = 2(6)$
$a_5 = 2(4)$
$a_6 = 2(1)$

Add

What if you were not given a definition of the function??????????? Making six rectangles from 0 to 12, how do we set up so to estimate the area?

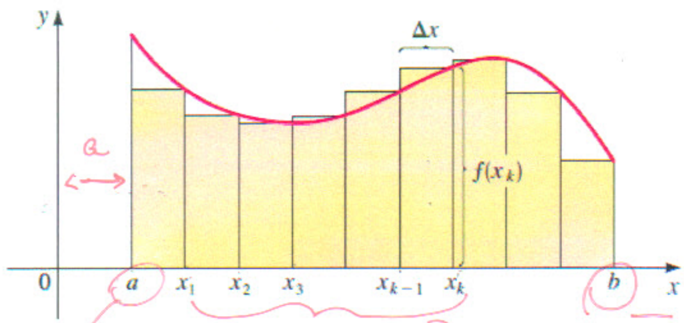
Here we will use right end points with 6 rectangles

$A = 2(8.8 + 8.1 + 7.4 + 6 + 4 + 1) = R_6$
$= 2(35.3) = \boxed{70.6}$

(an under estimate.)

Definition of Area

The area A of the region S that lies under the graph of a continuous function f is the limit of the sum of the areas of the approximating rectangles: ***use right endpoints***.



AREA=(height) x (width)
F(x) Δx

n rectangles

ending at "b"

Starting at "a"

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k\right)\Delta x$$

Δx **is the width** of an approximating rectangle,
x_k **is the right endpoint** of the kth **rectangle**
$f(x_k)$ **is its height.**

$\boldsymbol{n}$ *rectangles*
region from $x = a$ *to* $x = b$

width: $\boldsymbol{\Delta x = \frac{b-a}{n}}$

right endpoint: $\boldsymbol{x_k = a + k\Delta x}$

height: $\boldsymbol{f(x_k) = f(a + k\Delta x)}$

These Summation Formulas are an extension of the summation concepts studied in chapter 12:

Summation Formulas!!

these will be used in solving of our area problems

$$\sum_{k=1}^{n} c = nc$$

$$\sum_{k=1}^{n} k = \frac{n\left(n+1\right)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

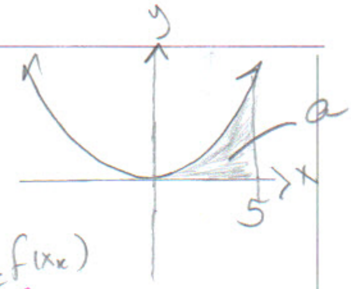
$$\sum_{k=1}^{n} k^3 = \frac{n^2\left(n+1\right)^2}{4}$$

Know!

Finding the Area under a Curve:

Find the area of the region that lies under

$y = x^2, 0 \le x \le 5$ $\quad$ area = (width)(height)



$f(x) = x^2$

width $= \Delta x = \dfrac{b-a}{n} = \dfrac{5-0}{n} = \dfrac{5}{n} = \Delta x$

height $= f(x_k) = f\left(\dfrac{5k}{n}\right) = \left(\dfrac{5k}{n}\right)^2 = \dfrac{25k^2}{n^2} = f(x_k)$

$x_k = a + k\Delta x$
$= 0 + k\left(\dfrac{5}{n}\right)$
$x_k = \dfrac{5k}{n}$

So

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \overbrace{\left(\frac{25k^2}{n^2}\right)}^{h} \times \overbrace{\left(\frac{5}{n}\right)}^{w}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{125}{n^3} k^2$$

Summation involves "k", so $\dfrac{125}{n^3}$ = constant

$$= \lim_{n\to\infty} \frac{125}{n^3} \sum_{k=1}^{n} k^2$$

Apply summation formula

$$= \lim_{n\to\infty} \frac{125}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

Rearrange

$$= \lim_{n\to\infty} \frac{125}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

Split the factors

$$= \lim_{n\to\infty} \frac{125}{6}\left(\frac{n}{n}\right)\left(\frac{n}{n} + \frac{1}{n}\right)\left(\frac{2n}{n} + \frac{1}{n}\right)$$

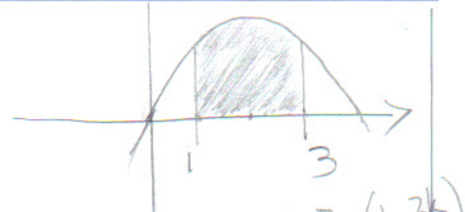
Evaluate for $n \to \infty$

$$= \frac{125}{\cancel{6}\,3}(\cancel{2}) = \boxed{\frac{125}{3}\text{ units}^2}$$

Finding an Area under a Curve

Find the area of the region that lies under

$y = 4x - x^2 \quad \underset{a}{1} \le x \le \underset{b}{3}$



$\Delta x = \frac{3-1}{n} = \underline{\frac{2}{n} = \Delta x}$

$x_k = 1 + k\left(\frac{2}{n}\right) = 1 + \frac{2k}{n} \Rightarrow f(x_k) = 4\left(1+\frac{2k}{n}\right) - \left(1+\frac{2k}{n}\right)^2 \left(1+\frac{2k}{n}\right)$

$= 4 + \frac{8k}{n} - \left(1 + \frac{4k}{n} + \frac{4k^2}{n^2}\right)$

$= 4 + \frac{8k}{n} - 1 - \frac{4k}{n} - \frac{4k^2}{n^2}$

$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x \qquad f(x_k) = 3 + \frac{4k}{n} - \frac{4k^2}{n^2}$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 + \frac{4k}{n} - \frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right)$ Distribute $\frac{2}{n}$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{6}{n} + \frac{8k}{n^2} - \frac{8k^2}{n^3}\right)$ Write as separate Σ terms

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{6}{n} + \sum_{k=1}^{n} \left(\frac{8}{n^2}\right)k - \sum_{k=1}^{n} \left(\frac{8}{n^3}\right)k^2$ ① pull out constant ② Evaluate Σ constant

$= \lim_{n\to\infty} \frac{6\cancel{n}}{\cancel{n}} + \frac{8}{n^2}\sum_{k=1}^{n} k - \frac{8}{n^3}\sum_{k=1}^{n} k^2$ Apply formulas for Σ

$= \lim_{n\to\infty} 6 + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$ Evaluate

$= \lim_{n\to\infty} 6 + \frac{8}{2}\left(\frac{n}{n}\right)\left(\frac{n}{n} + \cancelto{0}{\frac{1}{n}}\right) - \frac{8}{6}\left(\frac{n}{n}\right)\left(\frac{n}{n} + \cancelto{0}{\frac{1}{n}}\right)\left(\frac{2n}{n} + \cancelto{0}{\frac{1}{n}}\right)$ Simplify

$= 6 + 4 - \frac{8(2)}{\cancel{6}_3} = 10 - \frac{8}{3} = \frac{30}{3} - \frac{8}{3}$

$= \boxed{\frac{22}{3}\ \text{units}^2}$