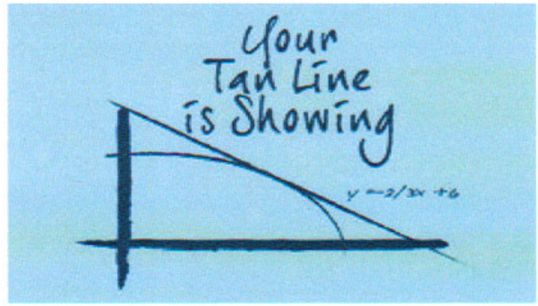
Precalculus

Lesson 14.3: The Tangent Problem: The Derivative

Mrs. Snow, Instructor

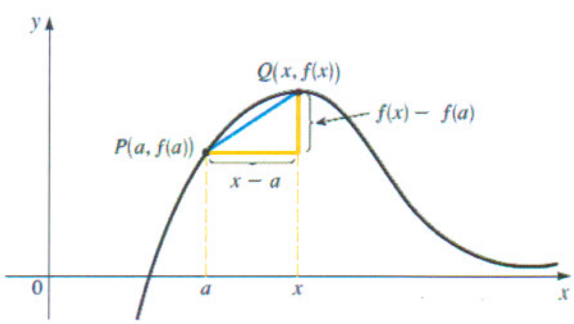


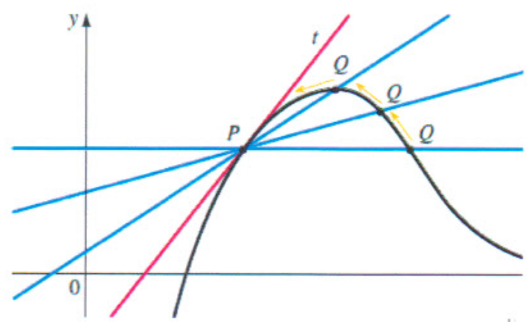
The slope of a line is used in application problems. Slope, is known as a rate of change, like a velocity. When the function is non-linear, the rate of change is not constant, so we can find the average velocity. What if we want (or maybe we don't want to but, have to .....) to find the exact velocity at an exact moment in time???? This is where the slope of a line tangent to a curve comes into play, and limits. Limits may be used to calculate the slope of a line that is tangent to a point on a curved graph. As a little foreshadowing, this tangent line problem and calculating its slope gave rise to the branch of calculus called *differential calculus* which dates back to the early to mid- 1600s!

If we think about our geometry definition of a tangent line, it is words to the effect of: A line that just touches a curve at one point, without cutting across it. In calculus we get the following:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$\boldsymbol{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$





Find an equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point (3,1).

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$(a, f(a))$

$a = 3$

$f(3) = 1$

$$= \lim_{x \to 3} \frac{\left(\frac{3}{x} - 1\right)}{(x-3)} \cdot \frac{(x)}{(x)}$$

Multiply by "1" & distribute

$$= \lim_{x \to 3} \frac{3 - x}{(x-3)(x)}$$

Almost the same, − factor out (−1)

$$= \lim_{x \to 3} \frac{-1\,\cancel{(x-3)}}{(x)\cancel{(x-3)}}$$

direct subst.

3

$$\underline{m = \frac{-1}{3}} \text{ at the point } (3,1)$$

$$y = mx + b \Rightarrow 1 = \frac{-1}{\cancel{3}}(\cancel{3}) + b$$

$$1 = -1 + b$$

$$\underline{\underline{2 = b}}$$

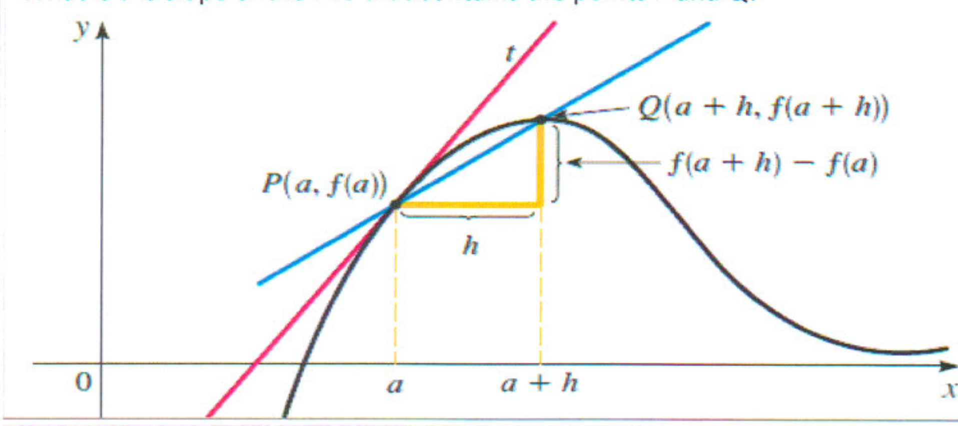
Equation of line tangent at (3,1)

$$\boxed{y = \frac{-1}{3}x + 2}$$

Another way to calculate slope (and sometimes easier to use) is as follows:

$$\boldsymbol{slope = \frac{f(a+h) - f(a)}{h}}$$

What is the slope of the line that contains the points P and Q?



$$\boldsymbol{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Finding a Tangent Line

Find an equation of the tangent line to the curve $y = x^3 - 2x + 3$ at the point (1,2).

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim \frac{f(1+h)-2}{h} \qquad (a, f(a))$$

$$= \lim_{h\to 0} \frac{(1+h)^3 - 2(1+h) + 3 - 2}{h} =$$

use binomial Thm OR Pascal's Δ.

$$= \lim_{h\to 0} \frac{\cancel{1} + 3h + 3h^2 + h^3 - \cancel{2} - 2h + \cancel{3} - \cancel{2}}{h} =$$

Should simplify such that there are only "h" terms

$$\lim_{h\to 0} \frac{h + 3h^2 + h^3}{h} = \lim_{h\to 0} \frac{\cancel{h}(1 + \cancel{3h}^{0} + \cancel{h^2}^{0})}{\cancel{h}} = 1 = m$$

Direct Subst.

$y = mx + b$

$2 = 1(1) + b$

$1 = b$

$$\boxed{y = x + 1}$$

Ok, so we are looking at tangent lines and their respective slopes, we also know slope as *rate of change.* This is a very important concept in calculus and applications in science and engineering as many problems deal with velocities and rates of change. Because this type of limit occurs so widely, it is given a special name and notation.

Definition of a Derivative

The derivative of a function is the slope of the tangent line…so…

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Finding a Derivative at a Point

Find the derivative of the function $f(x) = 5x^2 + 3x - 1$ at the number $\boxed{2. = a}$

$f(2) = 5(2^2) + 3(2) - 1$
$= 20 + 6 - 1 = \boxed{25 = f(a)}$

$$f'(a) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\lim_{h \to 0} \frac{5(2+h)^2 + 3(2+h) - 1 - 25}{h}$$

$$\lim_{h \to 0} \frac{5(4 + 4h + h^2) + 6 + 3h - 1 - 25}{h}$$

$$\lim_{h \to 0} \frac{\cancel{20} + 20h + 5h^2 + \cancel{6} + 3h - \cancel{1} - \cancel{25}}{h}$$

$$\lim_{h \to 0} \frac{23h + 5h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(23 + 5h)}{\cancel{h}} = 23 = m$$

SO: The line that is tangent to $f(x)$ at $x = 2$ has a slope of 23.

Let $f(x) = \sqrt{x}$ $\quad f(a) = \sqrt{a}$

Find the derivatives: $f'(a), f'(1), f'(4)$, and ~~$f''(9)$~~ $f'(9)$

use conjugates

$$f'(a) = \lim_{x \to a} \frac{(\sqrt{x} - \sqrt{a})}{(x - a)} \cdot \frac{(\sqrt{x} + \sqrt{a})}{(\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \to a} \frac{\cancel{(x - a)}\ 1}{\cancel{(x - a)}(\sqrt{x} + \sqrt{a})}$$

$$= \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}} = f'(a)$$

find $f'(1)$, $f'(4)$, $f'(9)$

$$\frac{1}{2\sqrt{1}} = f'(1) = \frac{1}{2}$$

$$\frac{1}{2\sqrt{4}} = \frac{1}{4} = f'(4)$$

$$\frac{1}{2\sqrt{9}} = \frac{1}{6} = f'(9)$$

Using the other Formulas:

problems from previous page

$f(x) = 5x^2 + 3x - 1$ at $a = 2$

$f(a) = 25$

$$f'(a) = \lim_{x \to 2} \frac{5x^2 + 3x - 1 - (25)}{x - 2}$$

$$= \lim_{x \to 2} \frac{5x^2 + 3x - 26}{x - 2}$$

$$= \lim_{x \to 2} \frac{(5x + 13)\cancel{(x-2)}}{\cancel{(x-2)}}$$

(hint: one of the factors should be equal to denominator, if math is correct)

$$= 2(5) + 13 = \underline{\underline{23}} =$$

$f(x) = \sqrt{x}$ $\quad f(a) = \sqrt{a}$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(\sqrt{a+h} - \sqrt{a})}{(h)} \cdot \frac{(\sqrt{a+h} + \sqrt{a})}{(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \to 0} \frac{\cancel{a} + h - \cancel{a}}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{a + \underset{\uparrow 0}{h}} + \sqrt{a})}$$

$$= \frac{1}{\sqrt{a} + \sqrt{a}} = \underline{\underline{\frac{1}{2\sqrt{a}}}} = f'(a)$$

Instantaneous Rate of Change:
Suppose we consider the average rate of change over smaller and smaller intervals by letting x approach a. The limit of these average rates of change is called the **instantaneous rate of change.** In the special case where x= $t = time$ and $s = f(t) = displacement\ at\ time\ t$ of an object traveling in a straight line, we call the instantaneous rate of change **instantaneous velocity.**

If an object is dropped from a height of 3000 ft, its distance above the ground (in feet) after t seconds is given by $h(t) = 3000 - 16t^2$. Find the object's instantaneous velocity after 4 seconds.

$h(4) = 3000 - 16(4^2)$

$h(4) = 2744$ ft at $t = 4$ sec

$$h' = \lim_{t\to 4} \frac{3000 - 16t^2 - 2744}{t-4}$$

factor?!

hint: one factor should be $(t-4)$:)

$$= \lim_{t\to 4} \frac{-16t^2 + 256}{t-4}$$

$$= \lim_{t\to 4} \frac{-16(t^2-16)}{(t-4)} = \lim_{t\to 4} \frac{-16(t+4)\cancel{(t-4)}}{\cancel{(t-4)}}$$

$4 =$

$$= -16(4+4) = -16(8) = \boxed{-128 \text{ ft/sec}}$$

OR

$f(4+h) - f(4)$

$$h' = \lim_{h\to 0} \frac{3000 - 16(4+h)^2 - 2744}{h}$$

$$= \lim_{h\to 0} \frac{3000 - 16(16 + 8h + h^2) - 2744}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3000} - \cancel{256} - 128h - 16h^2 - \cancel{2744}}{h}$$

$$= \lim_{h\to 0} \frac{-128h - 16h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(-128 - \cancel{16h})}{\cancel{h}}$$

0

$$= \boxed{-128 \text{ ft/sec}}$$