Precalculus
Lesson 12.5: The Binomial Theorem
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An expression with two terms is called a **binomial** for example $a + b$ is a binomial. It is an easy enough process to square this binomial or to cube it, but expanding this binomial by a higher degree or multiplying it out more times, will quickly get tedious. Looking at the binomial expansion of $a + b$ for the first five degrees we should see a pattern:

4c. Expand $(a+b)^n$

$(a+b)^n$
$= (a \;+\; b)^n$
$= (a \quad + \quad b)^n$
$= (a \qquad + \qquad b)^n$

Very funny Bob.

Expanding $(a + b)^n$

$(a+b)^1 = a+b$ degree 1 — 2 terms

$(a+b)^2 = a^2 + 2ab + b^2$ degree 2 - 3 terms

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ deg 3 - 4 terms

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ deg 4 - 5 terms

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$a^5b^0 \quad a^4b^1 \quad a^3b^2 \quad a^2b^3 \quad a^1b^4 \quad a^0b^5$

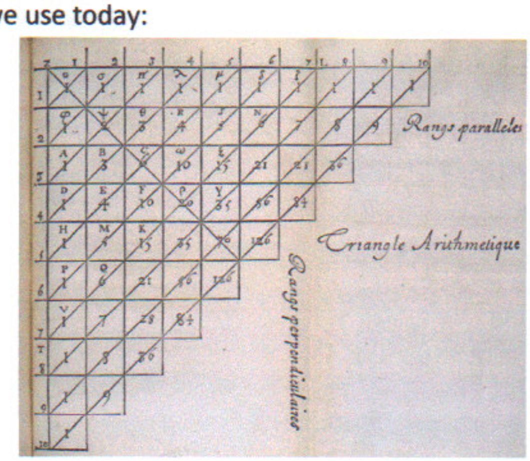
What is the pattern?

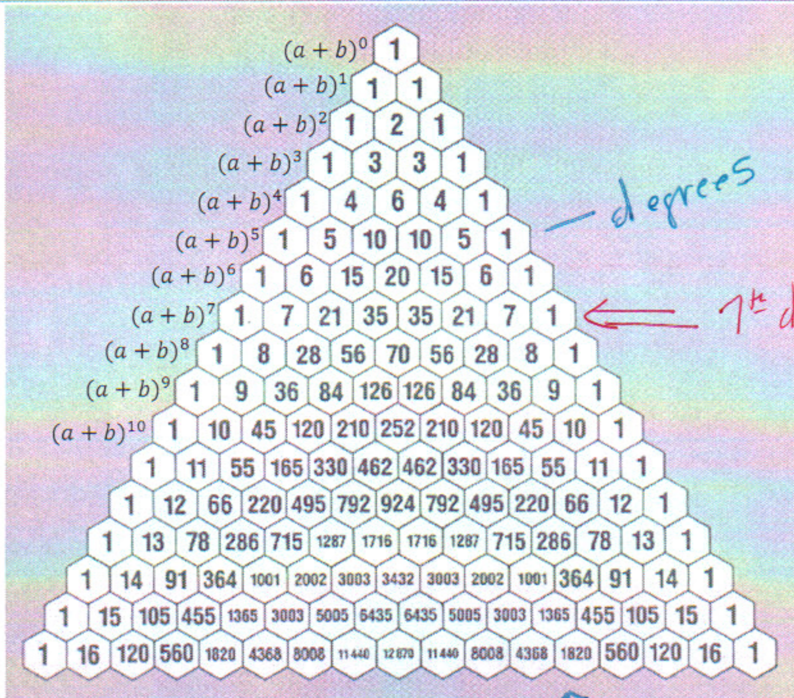
$(a + b)^n$

1. There are $n + 1$ terms, the first being $\boldsymbol{a^n}$ and the last is $\boldsymbol{b^n}$.
2. The exponents of $\boldsymbol{a}$ decrease by 1 from term to term while the exponents of $\boldsymbol{b}$ increase by one
3. The sum of the exponents of $\boldsymbol{a}$ ***and*** $\boldsymbol{b}$ in each term is $\boldsymbol{n}$

The pattern that is present in binomial expansion has been known for centuries. Blaise Pascal organized it into a triangular format that has become known as Pascal's Triangle. Below are both his original version and what we use today:

A bit of history.....





— degrees

$\Leftarrow$ 7^{th} dg.

note symmetry on coefficients

Using Pascal's Triangle to expand binomials

Expand $(a+b)^7$ coefficients

1 7 21 35 35 21 7 1

$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

$a^7 \longrightarrow a^0$
$b^0 \longrightarrow b^7$ exponents

$(2-3x)^5$ $a=2$ $b=-3x$ coefficients: 1 −5 − 10 − 10 − 5 − 1

$(1)(2)^5 + 5(2^4)(-3x) + 10(2^3)(-3x)^2 + 10(2^2)(-3x)^3 + 5(2^1)(-3x)^4 + (-3x)^5$

$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$

So: when $(a-b)^n \rightarrow$ alternating $\pm$ pattern too.

Pascal's Triangle is pretty slick for binomial expansions with relatively small values of n. For very large exponents, we need a more efficient way to calculate the coefficients. Pascal's Triangle is recursive in that to find the 100th row, we need the 99th row. So to come up with a process, we will need to use **factorials** that we studied in 12.1.

Binomial Coefficients

If j and n are integers with $0 \le j \le n$, the symbol $\binom{n}{j}$ is defined as

$$\boxed{\binom{n}{j} = \frac{n!}{j!(n-j)!}}$$

Calculate the binomial coefficients

$$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9!}{4!\,5!} = \frac{\cancel{9}^{3}\cdot\cancel{8}\cdot 7\cdot 6\cdot\cancel{5!}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{5!}}$$

$$= 3\cdot 7\cdot 6 = \underline{126}$$

42×3

$$\binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{100!}{3!\,97!} = \frac{\cancel{100}^{50}\cdot\cancel{99}^{33}\cdot 98\cdot\cancel{97!}}{\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{97!}}$$

$$= 50\cdot 33\cdot 98$$

$$= \underline{161700}$$

This helps up because the values of Pascal's Triangle are in fact binomial coefficients!

n = exponent

j = order of term minus 1

Notice:
$n-j$ = exponent of the 1st term

$$\binom{0}{0} \qquad (a+b)^0$$

$$\binom{1}{0}\ \binom{1}{1} \qquad (a+b)^1$$

$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \qquad (a+b)^2$$

$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \qquad (a+b)^3$$

$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \qquad (a+b)^4$$

exp's → 5 4 3 2 1 0

$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \qquad (a+b)^5$$

$$\cdots$$

$$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \cdots\ \binom{n}{n-1}\ \binom{n}{n} \qquad (a+b)^n$$

Binomial Theorem

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$\boxed{\begin{aligned}(x+a)^n &= \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n \\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j\end{aligned}}$$

Use the Binomial Theorem to expand the following:

$(x+y)^4$

$$\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$\boxed{= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}$$

$$\binom{4}{0} = \frac{4!}{0!4!} = 1 \quad \binom{4}{1} = \frac{4!}{1!3!} = \frac{4\cdot\cancel{3!}}{\cancel{3!}} = 4 \quad \binom{4}{2} = \frac{4!}{2!} = \frac{4!}{2!2!} = \frac{4\cdot 3\cdot\cancel{2!}}{2\cdot\cancel{2!}} = 6$$

$$\binom{4}{3} = \frac{4!}{3!} = \frac{4!}{3!1!} = \frac{4\cdot\cancel{3!}}{\cancel{3!}} = 4 \quad \binom{4}{4} = \frac{4!}{0!4!} = 1$$

$(2y-3)^4$

Thank you! We don't need to calculate the factors as they are the same as previous example!!

$a = 2y$
$b = -3$

$$1(2y)^4 + 4(2y)^3(-3) + 6(2y)^2(-3)^2 + 4(2y)(-3)^3 + (-3)^4 =$$

$$\underline{16y^4 - 96y^3 + 216y^2 - 216y + 81}$$

The Binomial theorem may be used to find a particular term of a binomial expansion:

Based on the expansion of $(x + a)^n$, the term containing x^j is

Use to find a specific term with a given exponent.

$$\binom{n}{n-j} a^{n-j} x^j \quad (3)$$

Find the find the coefficient of y^8 in the expansion of $(2y + 3)^{10}$ $n = 10$

to find term that has y^8:

$n = 10$
$j = 8$

so $\binom{n}{n-j} = \binom{10}{10-8} = \binom{10}{2} = \frac{10!}{2!\,8!} = \frac{\cancel{10}^{5} \cdot 9 \cdot \cancel{8!}}{\cancel{2} \cdot \cancel{8!}} = \underline{45}$

term with y is $2y \Rightarrow$ for y^8 $(45)(2y)^8(3)^2$

exponents $8+2=10$

$= \underline{103680y^8}$

Find the 6^{th} term in the expansion of $(x + 2)^9$

n = exponent
j = order of term minus 1
$6-1=5$

$\binom{n}{j} = \binom{9}{5} = \frac{\cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 3 \cdot 7 \cdot 6 = 126$

What is the exponent of x for term 6?
$n - j = 9 - 5 = 4 \Leftrightarrow x^4$

6^{th} term $=$ (coefficient)$(x^?)(2^?)$
$= (126)(x^4)(2^5)$
$= \boxed{4032x^4}$

remember exponents add up to equal 9