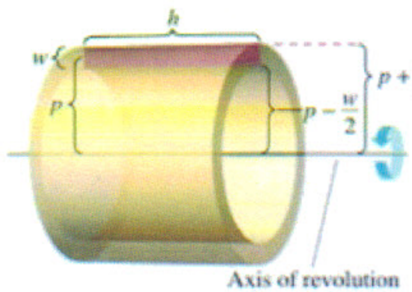
Calculus
Lesson 7.3: Volume: The Shell Method
Mrs. Snow, Instructor

In this section, you will study an alternative method for finding the volume of a solid of revolution. This method is called the shell method because it uses cylindrical shells. There are advantages and disadvantages between the shell and disk methods. More to come on this topic.



Given a rectangle of
width $= w$,
height$= h$
and the **distance** between the axis of revolution and center of the rectangle $= p$

When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness= w.

To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because p is the average radius of the shell:

$$outer\ radius\ = p + \left(\frac{w}{2}\right)$$
$$inner\ radius\ = p - \left(\frac{w}{2}\right)$$

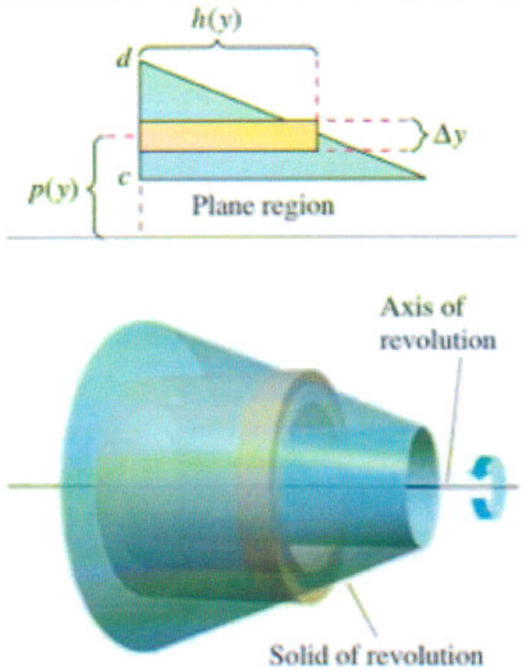
So the volume of the shell is:

$$Volume\ of\ shell = (volume\ of\ cylinder) - (volume\ of\ hole)$$

$$= \pi\left(p + \left(\frac{w}{2}\right)\right)^2 h - \pi\left(p - \left(\frac{w}{2}\right)\right)^2 h$$

$$= 2\pi phw$$

$$= 2\pi(average\ radius)(height)(thickness)$$



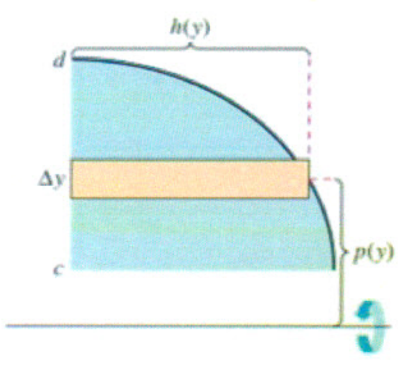
You can use this formula to find the volume of a solid of revolution. Assume that the plane region in the figure to the left is revolved about a line to form the indicated solid. If you consider a horizontal rectangle of width Δy, then, as the plane region is revolved about a line parallel to the x-axis, the rectangle generates a representative shell whose volume is:

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$

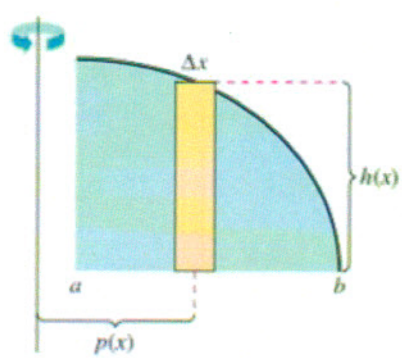
THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 7.29.

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$



Horizontal axis of revolution

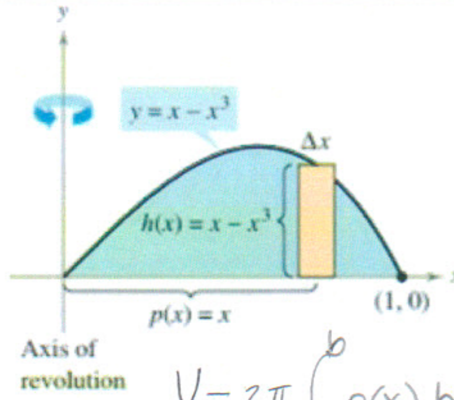


Vertical axis of revolution

Using the Shell Method to find Volume

- Find the volume of the solid of revolution formed by revolving the region bounded by y and the x-axis about the y-axis.*

$y = x - x^3 \qquad 0 \le x \le 1$



* Vertical axis → y-axis so use vertical rectangle.

→ Δx indicates that x is the variable of integration.

$$V = 2\pi \int_a^b p(x)\, h(x)\, dx$$

$h(x) = x - x^2$

$P(x)$ = distance to center of rectangle from axis of rev.

$P(x) = x$.

$$V = 2\pi \int_0^1 x\,(x - x^3)\,dx$$

$$= 2\pi \int_0^1 x^2 - x^4\,dx$$

$$= 2\pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right)\Bigg|_0^1$$

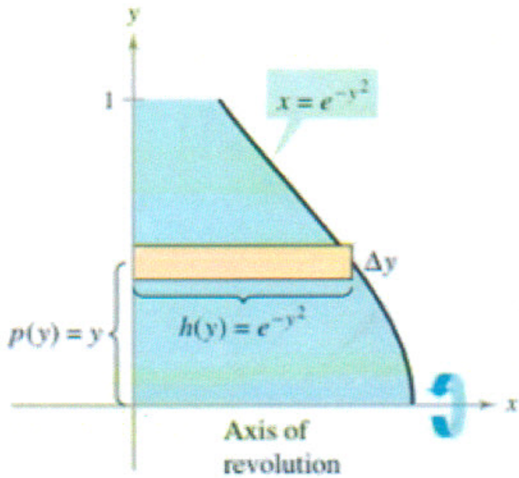
$$= (2\pi)\left(\frac{1}{3} - \frac{1}{5} - (0) \right)$$

$$(2\pi)\left(\frac{5}{15} - \frac{3}{15} \right) = 2\pi\left(\frac{2}{15}\right) = \boxed{\frac{4\pi}{15}}$$

Using the Shell Method to Find Volume

- Find the volume of the solid of revolution formed by revolving the region bounded by the graph of x and the y-axis about the x-axis. *

$x = e^{-y^2}$ $\qquad 0 \le y \le 1$



* Axis of revolution is horizontal: use a horizontal rectangle.

→ Height of rectangle is Δy, so y is variable of integration

→ Distance from axis of revolution to center of rectangle is $p(y) = y$

$V = 2\pi \int_c^d p(y)\, h(y)\, dy$ and: $h(y) = x = e^{-y^2}$

$= 2\pi \int_0^1 y e^{-y^2} dy =$

$u = -y^2$
$du = -2y\,dy$
$-\frac{1}{2} du = y\,dy$

$= (\cancel{2}\pi)\left(\frac{-1}{\cancel{2}}\right) \int_0^1 e^u du =$

$= (-\pi) e^u = (-\pi)\left(e^{-y^2}\right)\Big|_0^1$

$= (-\pi)\left(e^{-(1^2)} - e^0\right)$

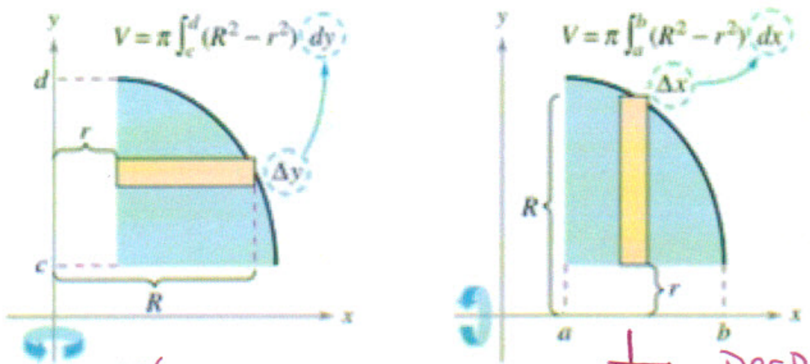
$= (-\pi)\left(\frac{1}{e} - 1\right)$ OR $\boxed{\pi\left(1 - \frac{1}{e}\right)}$

Comparison of the Disk and Shell Methods

The disk and shell methods can be distinguished as follows.

- For the disk method, the representative rectangle is always ***perpendicular*** to the axis of revolution.
- For the shell method, the representative rectangle is always ***parallel*** to the axis of revolution, as shown in the figures below.

Disk Method:



$\perp$ perpendicular

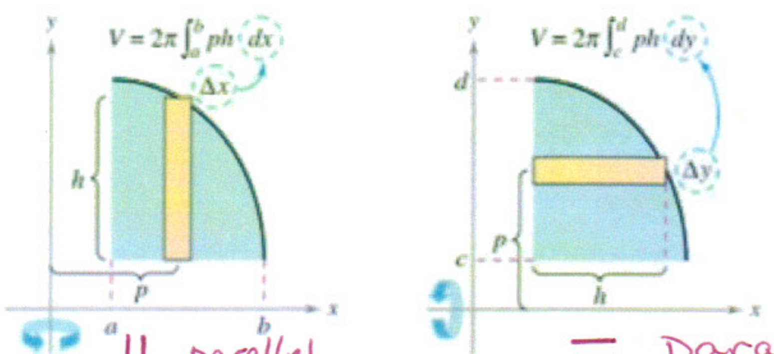
Vertical axis of revolution

$\perp$ perpendicular

Horizontal axis of revolution

Disk method: Representative rectangle is perpendicular to the axis of revolution.

Shell Method:



$\parallel$ parallel

Vertical axis of revolution

$=$ parallel

Horizontal axis of revolution

Shell method: Representative rectangle is parallel to the axis of revolution.

Often, one method is more convenient to use than the other. The following example illustrates a case in which the shell method is preferable. Here using the disk method, we would need two integrals to find the volume.

Shell Method Preferable

- Find the volume of the solid formed by revolving the region bounded by the following graphs about the y-axis.

$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$

$$y = 2\pi \int_a^b p(x)\, h(x)\, dx$$

$$= 2\pi \int_0^1 x(x^2+1)\,dx$$

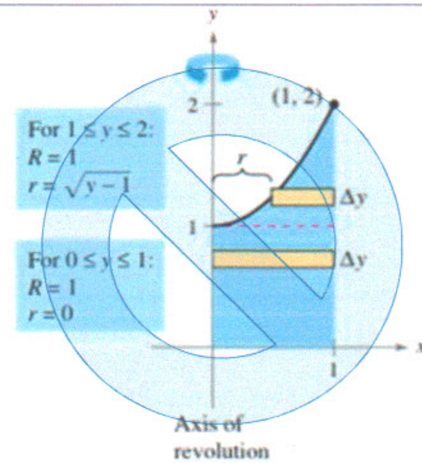
$$= 2\pi \int_0^1 x^3 + x \; dx =$$

$$= (2\pi)\left[\frac{x^4}{4} + \frac{x^2}{2}\right]\Bigg|_0^1$$

$$= (2\pi)\left(\frac{1}{4} + \frac{1}{2} - (0)\right)$$

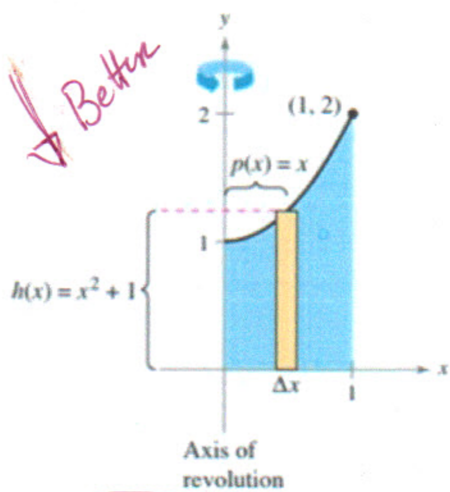
$$= 2\pi\left(\frac{1}{4} + \frac{2}{4}\right)$$

$$\cancel{2}\pi\left(\frac{3}{\cancel{4}_{2}}\right) = \boxed{\frac{3\pi}{2}}$$



(a) Disk method

Better



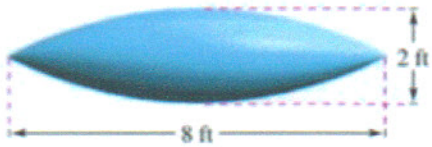
(b) Shell method

Here the **Disk Method is preferable** using only one integral compared the Shell Method which would require two integrals.

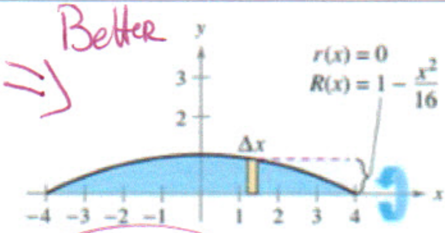
Volume of a Pontoon: Disk Method is preferable

- A pontoon is designed by rotating the graph about the x-axis, where x and y are measured in feet. Find the volume of the pontoon.

$y = 1 - \frac{x^2}{16}, \qquad -4 \le x \le 4$

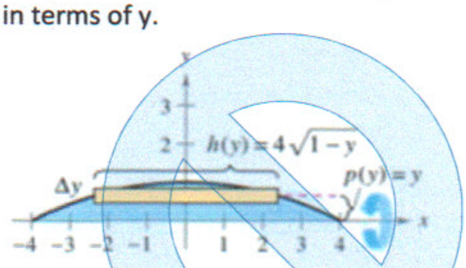


Better



(a) Disk method

Here you will have to solve the equation for x in terms of y.



(b) Shell method

Disk Volume:

$$V = \pi \int_a^b (R(x))^2\,dx =$$

$$= \pi \int_{-4}^{4} \left(1 - \frac{x^2}{16}\right)^2 dx$$

$$= \pi \int_{-4}^{4} 1 - \frac{x^2}{8} + \frac{x^4}{256}\,dx$$

$$= (\pi)\left(x - \frac{x^3}{3(8)} + \frac{x^5}{5(256)}\right)\Bigg|_{-4}^{4}$$

$$= (\pi)\left(4 - \frac{\cancel{64}\,8}{3(\cancel{8})} + \frac{4(\cancel{256})}{5(\cancel{256})} - \left(-4 - \frac{(-4)^3}{3(8)} + \frac{(-4)^5}{5(256)}\right)\right)$$

$$= \pi\left(4 - \frac{8}{3} + \frac{4}{5}\right) - \left(-4 + \frac{\cancel{64}\,8}{3(\cancel{8})} - \frac{4(\cancel{256})}{5(\cancel{256})}\right)$$

$$= \pi\left(\cancel{4} - \frac{8}{3} + \frac{4}{5} + \cancel{4} - \frac{8}{3} + \frac{4}{5}\right)$$

$$= \pi\left(8 - \frac{16\cdot 5}{3\cdot 5} + \frac{8}{5}\right)$$

$$= \pi\left(\frac{120}{15} - \frac{80}{15} + \frac{24}{15}\right) =$$

$$\boxed{\frac{64\pi}{15}}$$

$$= \left(1 - \frac{x^2}{16}\right)\left(1 - \frac{x^2}{16}\right)$$

$$= 1 - \frac{2x^2}{16} + \frac{x^4}{256}$$

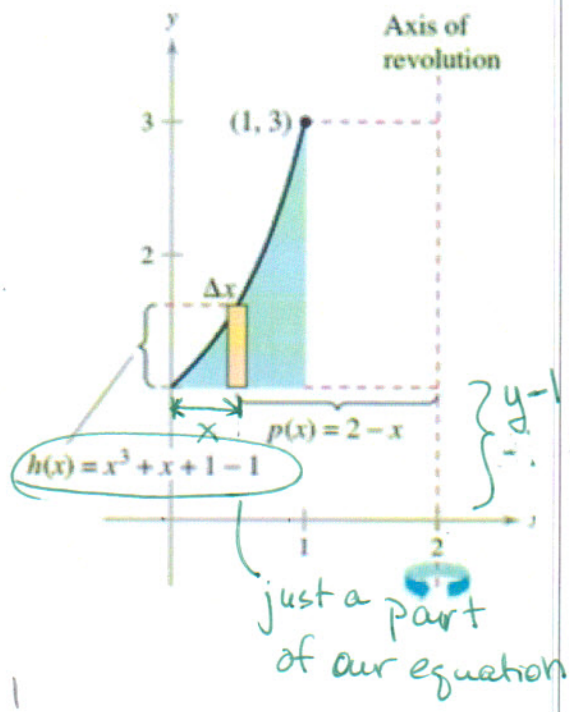
$$= 1 - x^2/8 + x^4/256$$

Sometimes, solving for x is very difficult (or even impossible). In such cases you must use a vertical rectangle (of width Δx), thus making x the variable of integration. The position (horizontal or vertical) of the axis of revolution then determines the method to be used.

Shell Method Necessary

Find the volume of the solid formed by revolving the region bounded by the graphs about the line x=2

$$y = x^3 + x + 1, \quad y = 1, \quad \text{and} \quad x = 1$$



x = ? ARRGH! so we have to integrate with respect to x.

$$V = 2\pi \int_a^b p(x)\, h(x)\, dx$$

$$= 2\pi \int_0^1 (2-x)(x^3+x+\cancel{1}-\cancel{1})\, dx$$

$$= 2\pi \int_0^1 (-x^4+2x^3-x^2+2x)\, dx$$

$$= 2\pi \left(-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2 \right) \Big|_0^1$$

$$2\pi \left[-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1 - (0) \right]$$

$$2\pi \left[-\frac{6}{30} + \frac{15}{30} - \frac{10}{30} + \frac{30}{30} \right]$$

$$2\pi \left(\frac{29}{30} \right) = \boxed{\frac{29\pi}{15}}$$

$$(2-x)(x^3+x)$$

$$2x^3+2x-x^4-x^2$$