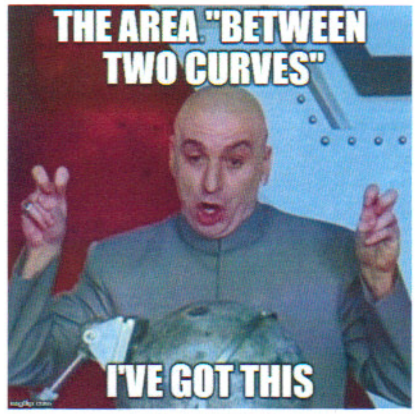
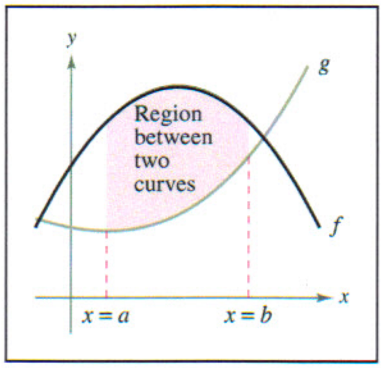
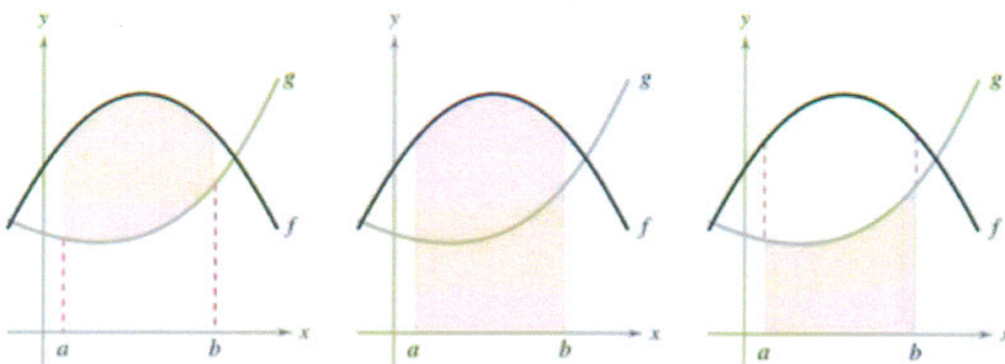
Calculus
Lesson 7.1: Area of a Region Between Two Curves
Mrs. Snow, Instructor



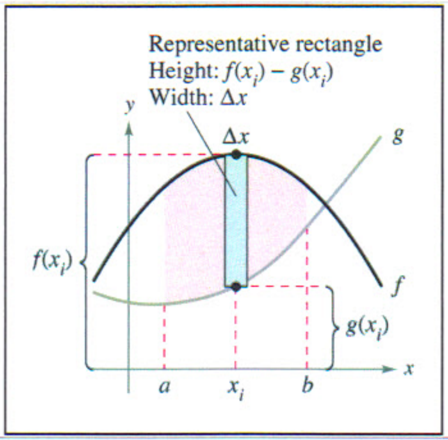
We can extend the application of definite integrals from the area of a region **under a curve** to the area of a region **between two curves**.

Region Between Two Curves





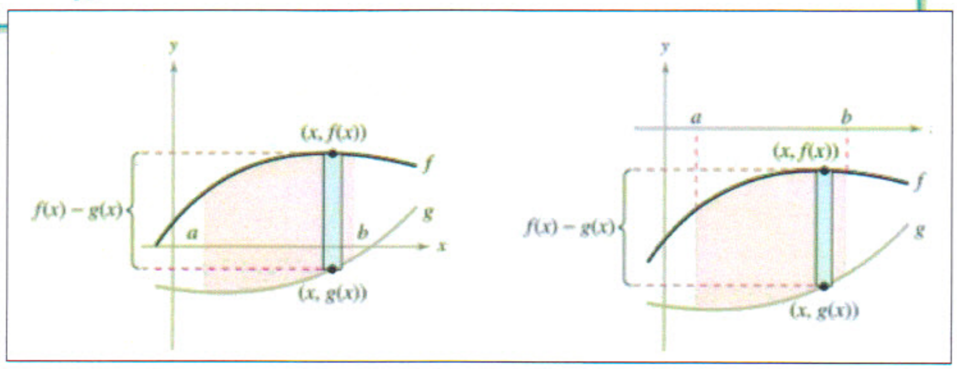
Area of region between f and g	=	Area of region under f	−	Area of region under g
$\int_a^b [f(x) - g(x)]\,dx$	=	$\int_a^b f(x)\,dx$	−	$\int_a^b g(x)\,dx$



AREA OF A REGION BETWEEN TWO CURVES

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$



Notice on the above graphs, the same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a, b]$.

Finding the Area of a Region Between Two Curves

- Find the area of the region bounded by the graphs of

$$y = x^2 + 2,\ y = -x,\ x = 0, \text{ and } x = 1$$

$\int$ greater function − Lesser function

$$\int_a^b f(x) - g(x)\, dx =$$

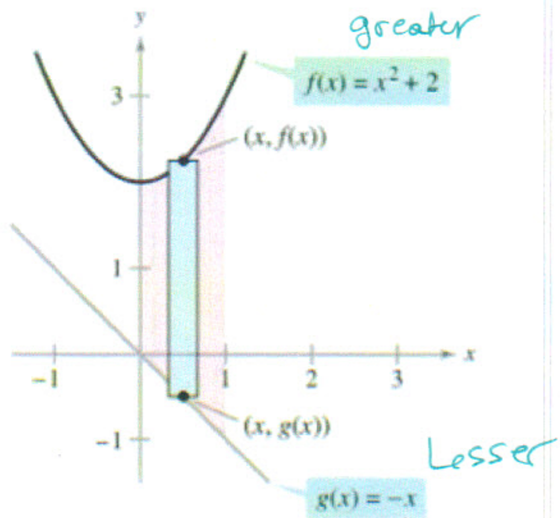
$$\int_0^1 x^2 + 2 - (-x)\, dx =$$

$$\int_0^1 x^2 + x + 2\, dx =$$

$$\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_0^1 =$$

$$\frac{1}{3} + \frac{1}{2} + 2 - (0)$$

$$\frac{2}{6} + \frac{3}{6} + \frac{12}{6} = \boxed{\frac{17}{6}}$$



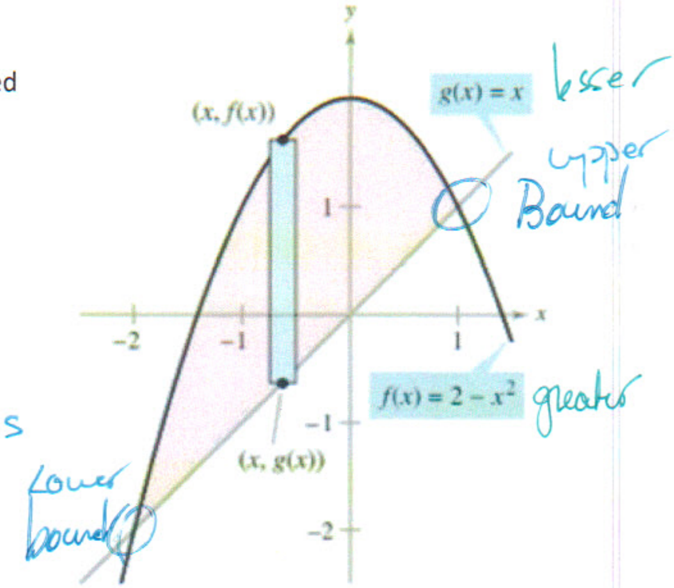
Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$

A Region Lying Between Two Intersecting Graphs

- Find the area of the region bounded by the graphs

$f(x)=2-x^2$ and $g(x)=x$

greater   Lesser



Region bounded by the graph of f and the graph of g

Intersection points are upper & Lower bounds

$2-x^2=x$

$0=x^2+x-2$

$0=(x+2)(x-1)$

$x=-2, 1$

$$\int_{-2}^{1} 2-x^2-(x)\,dx = \int_{-2}^{1} 2-x^2-x\,dx =$$

$$2x-\frac{x^3}{3}-\frac{x^2}{2}\Big|_{-2}^{1}$$

$$2-\frac{1}{3}-\frac{1}{2}-\left(2(-2)-\left(\frac{-8}{3}\right)-\frac{4}{2}\right)$$

$$2-\frac{1}{3}-\frac{1}{2}+4-\frac{8}{3}+2$$

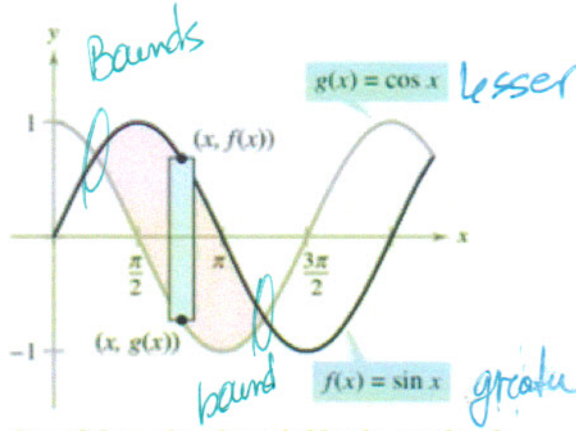
$$\frac{48}{6}-\frac{2}{6}-\frac{3}{6}-\frac{16}{6}=\frac{27}{6}=\boxed{\frac{9}{2}}$$

48
-21

A Region Lying Between Two Intersecting Graphs

- The sine and cosine curves intersect infinitely many times, bounding regions of equal area. Find the area of one of these regions.

Before we can find the area of one of the regions, we must find determine the upper and lower bounds.



One of the regions bounded by the graphs of the sine and cosine functions

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = \tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

greater − Lesser

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx = -\cos x - \sin x \Big|_{\pi/4}^{\frac{5\pi}{4}}$$

$$= -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

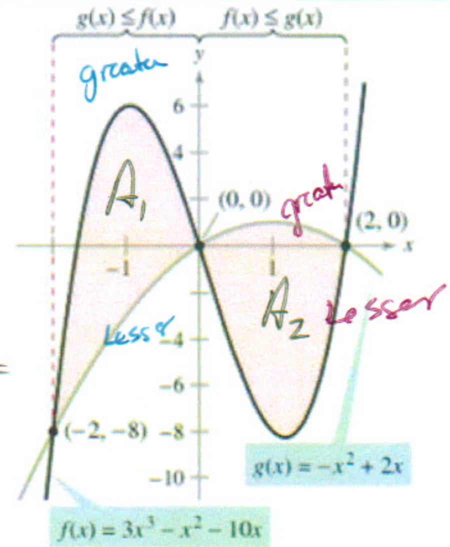
$$-\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = \boxed{2\sqrt{2}}$$

Curves that Intersect at More Than Two Points

- Find the area of the region between the graphs of

$f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$



On $[-2, 0]$, $g(x) \le f(x)$, and on $[0, 2]$, $f(x) \le g(x)$

Area 1:

$$A_1 = \int_{-2}^{0} 3x^3 - x^2 - 10x - (-x^2 + 2x)\,dx =$$

$$= \int_{-2}^{0} 3x^3 - \cancel{x^2} - 10x + \cancel{x^2} - 2x\,dx =$$

$$\int_{-2}^{0} 3x^3 - 12x\,dx =$$

$$\frac{3}{4}x^4 - 6x^2 \Big|_{-2}^{0} = 0 - \left(\frac{3}{\cancel{4}_1}(\cancel{16}^4) - 6(4)\right)$$

$$= -(12 - 24) = \boxed{12 = A_1}$$

Area 2:

$$\int_0^2 -x^2 + 2x - (3x^3 - x^2 - 10x)\,dx$$

$$= \int_0^2 -\cancel{x^2} + 2x - 3x^3 + \cancel{x^2} + 10x\,dc$$

$$= \int_0^2 12x - 3x^3\,dx = 6x^2 - \frac{3}{4}x^4 \Big|_0^2 = 24 - \frac{3}{4}(16) - (0)$$

$$= 24 - 12 = \boxed{12 = A_2}$$

$$\text{Total Area} = A_1 + A_2 = 12 + 12 = \boxed{24}$$

In this chapter we will be using representative rectangles in various applications of integration.

- ***A vertical rectangle (width of Δx) implies integration with respect to x.***
- ***A horizontal rectangle (width of Δy) implies integration with respect to y.***

Integration with respect to y.

Integrate w/ y bounds & dy

Lesser

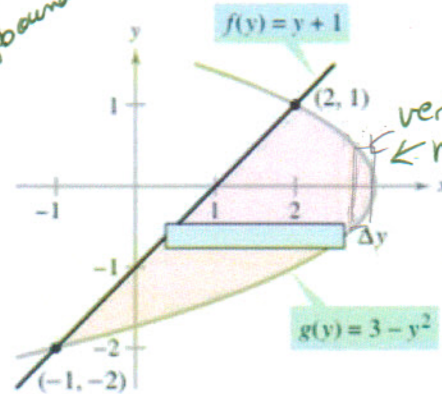
Horizontal Representative Rectangles

- Find the area of the region bounded by the graphs of

$x = 3 - y^2$ and $x = y + 1$ Lesser

greater

Horizonizontal rectangles: look at greatest x for the greater function



vertical not possible

greater

Horizontal rectangles (integration with respect to y)

Note that a vertical rectangle will not work - will not catch both curves.

$$\int_{-2}^{1} 3 - y^2 - (y+1)\,dy =$$

$$\int_{-2}^{1} 3 - y^2 - y - 1\,dy$$

$$\int_{-2}^{1} -y^2 - y + 2\,dy =$$

$$-\frac{y^3}{3} + \frac{y^2}{2} + 2y \Big|_{-2}^{1} =$$

$$-\frac{1}{3} - \frac{1}{2} + 2 - \left(\frac{8}{3} - 2 - 4\right)$$

$$-\frac{2}{6} - \frac{3}{6} + \cancel{2} - \frac{16}{6} + \cancel{2} + \cancel{4} + 8\left(\frac{6}{6}\right.$$

$$-\frac{21}{6} + \frac{48}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$$