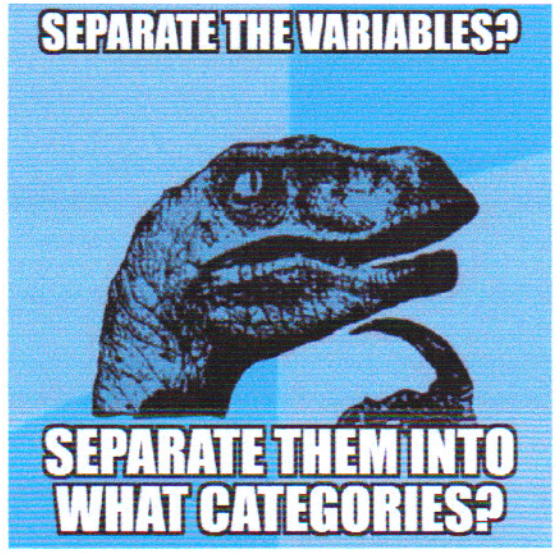
Calculus
Lesson 6.2 and 6.3: Differential Equations and Separation of Variables
Mrs. Snow, Instructor



Last lesson, we learned to analyze visually the solutions of differential equations using slope fields. In this section we will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called **separation of variables**.

Solving a differential equation through: Separation of Variables:

$y' = \frac{2x}{y} \Rightarrow (y)\frac{dy}{dx} = \frac{2x}{y}(y)$

$y\frac{dy}{dx}(dx) = 2x(dx)$

Separate like variables integrate

$y\,dy = 2x\,dx$

$\int y\,dy = \int 2x\,dx$

$\frac{y^2}{2} + c_1 - c_1 = x^2 + c_2 - c_1$ *

$(2)\frac{y^2}{2} = (x^2 + c)(2)$

$y^2 = 2x^2 + c$

or

$\boxed{y^2 - 2x^2 = c}$

*

Constants added/subtracted or multiplied still equal a constant!

$\frac{dy}{dx} = (xy)^2$

$\frac{dy}{dx} = x^2y^2$

$\int \frac{1}{y^2}\,dy = \int x^2\,dx$

$\int y^{-2}\,dy = \int x^2\,dx$

$-\frac{1}{y} = \frac{x^3}{3} + c$

$-1 = \left(\frac{1}{3}x^3 + c\right)y$

$\frac{-1}{\frac{1}{3}x^3 + c} = y$

$\boxed{\frac{-3}{x^3 + c} = y}$

Find the general solution:

$$\left(x^2+4\right)\frac{dy}{dx} = xy$$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx$$

$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{1}{u}\,du$$

$$\ln|y| = \frac{1}{2}\ln|u|$$

$$* \; e^{\ln|y|} = e^{\ln(x^2+4)^{1/2}+C}$$

$$|y| = \sqrt{x^2+4}\,(e^{c}) \; *$$

$$y = \pm e^{c}\sqrt{x^2+4}$$

as e^{c} = constant

$$\boxed{y = C\sqrt{x^2+4}}$$

* raise equation to powers

$e^{\log_e x} = x$ * $a^{m+n} = a^m a^n$

Finding a particular solution: given the initial condition of $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}\left(y^2-1\right)dy = 0$$

$$e^{-x^2}(y^2-1)\,dy = -xy\,dx$$

$$\frac{y^2-1}{y}\,dy = -xe^{x^2}\,dx$$

$$\int y - \frac{1}{y}\,dy = \int -xe^{x^2}\,dx$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$\frac{y^2}{2} - \ln|y| = \int -\frac{1}{2}e^{u}\,du$$

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{u}$$

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + C$$

Initial condition $y(0) = 1$

$$\frac{1^2}{2} - \cancel{\ln 1}^{\,0} = -\frac{1}{2}e^{0^2} + C$$

$$\frac{1}{2} = -\frac{1}{2} + C$$

$$1 = C$$

$$\boxed{\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + 1}$$

Find the equation of the curve that passes through the point (1,3) and has a slope of:

$$\frac{y}{x^2} = m \qquad \frac{dy}{dx} = \frac{y}{x^2}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx$$

$$e^{\ln|y|} = e^{-\frac{1}{x}+C}$$

e raised to a power

$$y = e^{-1/x}e^{c}$$

$$y = Ce^{-1/x} \quad \text{at } (1,3)$$

$$3 = Ce^{-1} \rightarrow 3e = C$$

exponent rules

$$y = 3e^{1}e^{-1/x}$$

$$1 - \frac{1}{x} = \frac{x}{x} - \frac{1}{x}$$

$$\boxed{y = 3e^{(x-1)/x}}$$