Precalculus

Lesson 11.5: Partial Fraction Decomposition

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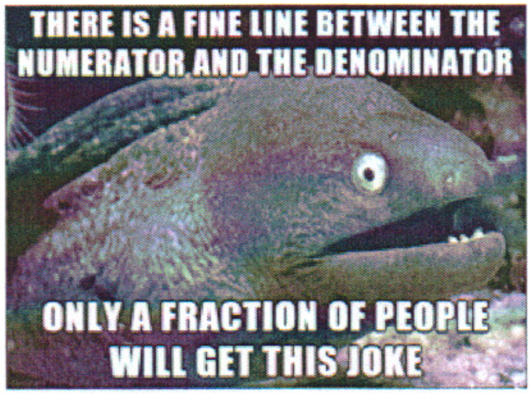

When we add fractions together we need to have a common denominator:

$$\frac{2}{x-1}+\frac{1}{2x+1}=\frac{2(2x+1)+(x-1)}{(x-1)(2x+1)}=\frac{3x+1}{2x^2-x-1}$$

Well, in some applications of algebra and calculus, we will need to reverse this process. Basically, we will need to take a fraction and express it as the sum of simpler fractions. This process is called **partial fraction decomposition.**

Non-repeated Linear Factors: The denominator have distinct linear factors, with no factor repeated.

Steps:

1. Factor the denominator into its linear factors.
2. Then write the expression as fractions with one factor for each of the denominator and variables for the numerators.
3. Clear denominators through multiplication
4. QUESTION: What can I let x be equal to so one of the expressions will go to zero??
5. Let x be equal to that number and solve for the remaining Numerator variable.
6. Now solve for the remaining Numerator variables.

$$\frac{x}{x^2-5x+6} = \frac{x \cancel{(x-3)(x-2)}}{\cancel{(x-3)(x-2)}} = \left(\frac{A}{x-3} + \frac{B}{x-2}\right)(x-3)(x-2)$$

distribute

$$x = A(x-2) + B(x-3)$$

Let $x = 2$

$$2 = A\underbrace{\cancel{(2-2)}}_{0} + B(2-3)$$

$$2 = 0 + B(-1)$$

$$2 = -B$$

$$\boxed{B = -2}$$

Let $x = 3$

$$3 = A(3-2) + B\underbrace{\cancel{(3-3)}}_{0}$$

$$\boxed{3 = A}$$

So: $$\frac{x}{x^2-5x+6} = \boxed{\frac{3}{x-3} + \frac{-2}{x-2}}$$

Repeated Linear Factors: Denominators have a linear factors that are repeated.

1. A linear factor in the denominator is raised to a power: Form fractions with increasing exponents of the factor until you reach the power of the factor.
2. Now multiply through by the common denominator.
3. What can I let x be equal to such that one of the expressions will go to zero?? Let x be equal to that number and solve for the remaining Numerator variable.
4. Now solve for the remaining Numerator variables.

$$\frac{x+2}{x^3-2x^2+x} = \frac{x+2}{x(x^2-2x+1)} = \frac{x+2}{(x)(x-1)^2}$$

$$\left(\frac{x+2}{x(x-1)(x-1)}\right)(x)(x-1)^2 = \left(\frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}\right)x(x-1)^2$$

$$x+2 = A(x-1)^2 + B(x)(x-1) + C(x)$$

Let $x=0$

$$2 = A(-1)^2$$

$$\underline{A=2}$$

Let $x=1$

$$1+2 = C$$

$$\underline{3=C}$$

to find B, we have A & C put these into equation and choose an "easy number" for x to solve for B.

$$x+2 = 2(x-1)^2 + B(x)(x-1) + 3x$$

Let $x=2$

$$4 = 2 + B(2)(2-1) + 3(2)$$

$$-4 = 2B$$

$$\underline{-2=B}$$

$\therefore$ Answer:

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} + \frac{-2}{(x-1)} + \frac{3}{(x-1)^2}$$

WOW!! You need to understand the set up:

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

clear out fractions

$$x^3-8 = Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2$$

let $x=0$

$$-8 = B(-1) \rightarrow \underline{B=8}$$

let $x=1$

$$1-8 = E \rightarrow \underline{E=-7}$$

Simplify Equation with B & E

$$x^3-8 = Ax(x-1)^3 + 8(x-1)^3 + Cx^2(x-1)^2 + Dx(x-1) - 7x^2$$

(expand & move left) move left

$$x^3-8-8(x^3-3x^2-3x-1)+7x = Ax(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1)$$

factor →

$$-7x^2 + \underbrace{31x^2 - 24x}_{\text{factor!}} = x(x-1)[A(x-1)^2 + Cx(x-1) + Dx]$$

$$\cancel{x(x-1)}(-7x+24) = \cancel{x(x-1)}[A(x-1)^2 + Cx(x-1) + Dx]$$

$$-7x+24 = A(x-1)^2 + Cx(x-1) + Dx$$

$x=0$

$\underline{24 = A}$

$x=1$

$\underline{17 = D}$

$$-7x+24 = 24(x-1)^2 + Cx(x-1) + 17x$$

let $x=2$

$$-14+24 = 24 + C(2) + 34$$

$$-48 = 2C$$

$\underline{-24 = C}$

Whew!

No worry! Not responsible for this!!

So: Ans:

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{24}{x} + \frac{8}{x^2} + \frac{-24}{x-1} + \frac{17}{(x-1)^2} + \frac{-7}{(x-1)^3}$$

Non-repeated Irreducible Quadratic Factors: The denominator contains a quadratic factor that is not factorable. Here the corresponding partial fraction decomposition will have the form:

$$\frac{P(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c}$$

Also remember these factoring formulas:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

1. Factor denominator if necessary
2. Clear fractions by multiplying each side with the left side's denominator.
3. Expand and combine like terms. This will lead now to building a system of equations
4. Equate coefficients to solve for the variables.

$$\frac{3x - 5}{x^3 - 1} = \left(\frac{3x-5}{(x-1)(x^2+x+1)}\right) = \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right)(x-1)(x^2+x+1)$$

$$3x - 5 = A(x^2+x+1) + (Bx+C)(x-1)$$

$\underline{x=1}$

$$3 - 5 = A(3) + \cancel{(Bx+C)(1-1)} \to 0$$

$$-2 = 3A \qquad \boxed{A = -\frac{2}{3}}$$

$\underline{x=0}$

$$-5 = -\frac{2}{3}(1) + (B(0)+C)(0-1)$$

$$-\frac{15}{3} + \frac{2}{3} = -C \quad \to \quad \boxed{C = \frac{13}{3}}$$

$$-\frac{13}{3} = -C$$

With A & C choose value for X & solve for B:

$\underline{x=-1}$

$$-8 = -\frac{2}{3}(1) + \left(B(-1) + \frac{13}{3}\right)(-2)$$

$$-8 = -\frac{2}{3} + 2B - \frac{26}{3}$$

$$-\frac{24}{3} + \frac{2}{3} + \frac{26}{3} = 2B$$

$$\left(\frac{1}{2}\right)\frac{4}{3} = 2B\left(\frac{1}{2}\right) \to \boxed{B = \frac{2}{3}}$$

Ans: $= \dfrac{-\frac{2}{3}}{x-1} + \dfrac{\frac{2}{3}x + \frac{13}{3}}{x^2+x+1}$

Non-repeated Irreducible Quadratic Factors: The denominator contains a quadratic factor that is not factorable. Here the corresponding partial fraction decomposition will have the form:

Repeat!

Just so you can see how to use system of equations to solve

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c}$$

Also remember these factoring formulas:

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

1. Factor denominator if necessary
2. Clear fractions by multiplying each side with the left side's denominator.
3. Expand and combine like terms. This will lead now to building a system of equations
4. Equate coefficients to solve for the variables.

$$\frac{3x-5}{x^3-1} = \frac{3x-5}{(x-1)(x^2+x+1)} = \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right)(x-1)(x^2+x+1)$$

use Linear rule

$$3x-5 = A(x^2+x+1) + (Bx+C)(x-1)$$

expand

$$0x^2+3x-5 = Ax^2+Ax+\boxed{A} + Bx^2-Bx+Cx\boxed{-C}$$

Coefficients of like terms are equal!

System of equations! :)

x^2: $0 = A+B$

x: $3 = A-B+C$ } Add:

constant: $-5 = A-C$

$$\begin{aligned} 0 &= A+B \\ 3 &= A-B+C \\ \hline 3 &= 2A+C \\ -5 &= A-C \\ \hline -2 &= 3A \\ -\tfrac{2}{3} &= A \end{aligned}$$

so

$B = +\frac{2}{3}$

$-5 = -\frac{2}{3} - C$

$-\frac{15}{3} + \frac{2}{3} = -C$

$C = \frac{13}{3}$

∴ Answer:

$$\frac{3x-5}{x^3-1} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}x+\frac{13}{3}}{x^2+x+1}$$

Repeated Quadratic Factors: Sometimes we will have a factorization that contains a quadratic factor that cannot be factored. The process is a combination of the previous two examples.

$$\frac{x^3+x^2}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

↑ Quadratic

$$x^3+x^2 = (Ax+B)(x^2+4) + Cx+D$$

Expand

$$x^3+x^2+0x+0 = Ax^3+4Ax+Bx^2+4B+Cx+D$$

Equate coefficients of like terms

$A=1$

$B=1$

$4A+C=0 \longrightarrow$ Substitute in A & B

$4+C=0$ or $C=-4$

$4B+D=0$

$4+D=0$ or $D=-4$

Answer:

$$\frac{x^3+x^2}{(x^2+4)^2} = \frac{x+1}{x^2+4} + \frac{-4x-4}{(x^2+4)^2}$$