Precalculus
Lesson 12.5: The Binomial Theorem
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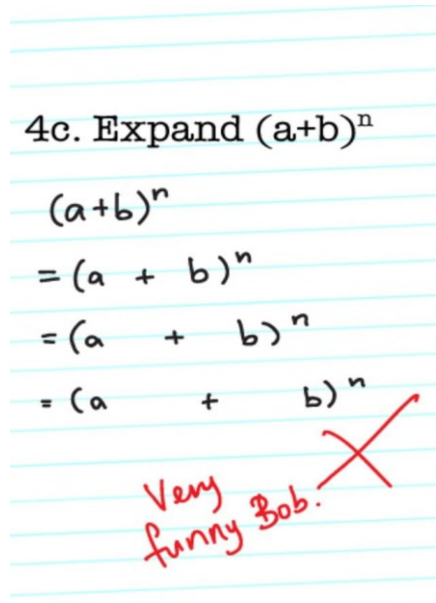
An expression with two terms is called a **binomial** for example $a+b$ is a binomial. It is an easy enough process to square this binomial or to cube it, but expanding this binomial by a higher degree or multiplying it out more times, will quickly get tedious. Looking at the binomial expansion of $a+b$ for the first five degrees we should see a pattern:



Expanding $(a+b)^n$

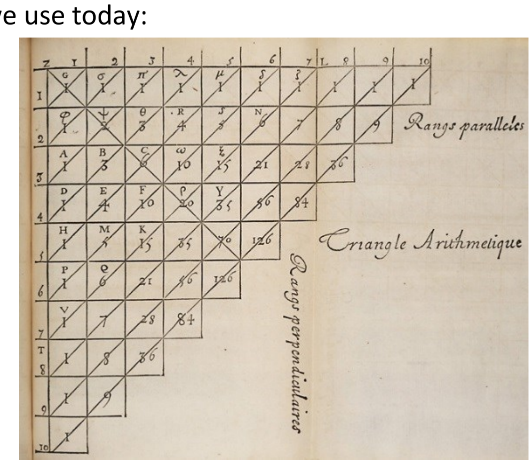
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

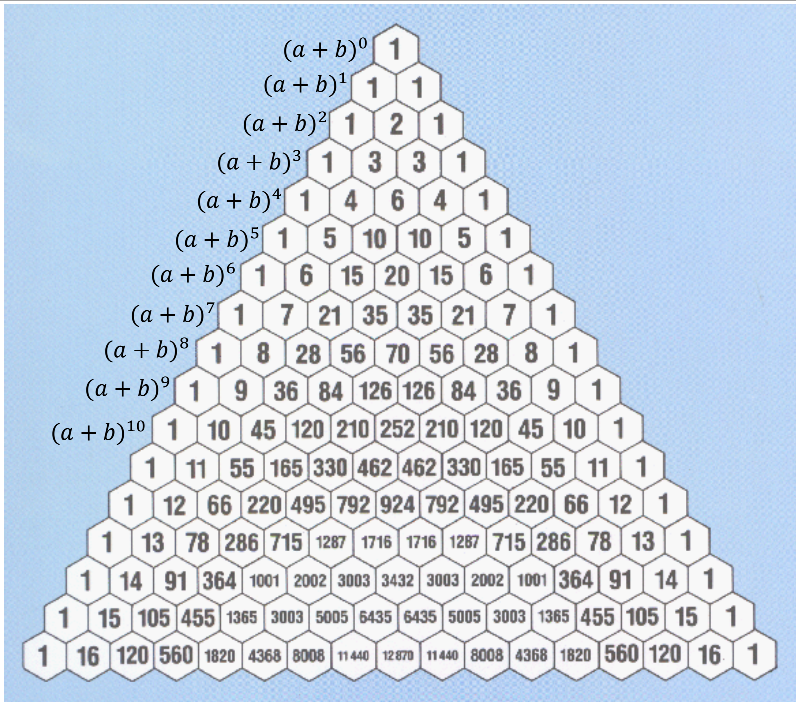
What is the pattern?

$$(a+b)^n$$

1. There are $n+1$ terms, the first being $\boldsymbol{a^n}$ and the last is $\boldsymbol{b^n}$.
2. The exponents of $\boldsymbol{a}$ decrease by 1 from term to term while the exponents of $\boldsymbol{b}$ increase by one
3. The sum of the exponents of $\boldsymbol{a\ and\ b}$ in each term is $\boldsymbol{n}$

The pattern that is present in binomial expansion has been known for centuries. Blaise Pascal organized it into a triangular format that has become known as Pascal's Triangle. Below are both his original version and what we use today:





Using Pascal's Triangle to expand binomials

Expand $(a+b)^7$
$(2-3x)^5$

Pascal's Triangle is pretty slick for binomial expansions with relatively small values of *n*. For very large exponents, we need a more efficient way to calculate the coefficients. Pascal's Triangle is recursive in that to find the 100th row, we need the 99th row. So to come up with a process, we will need to use **factorials** that we studied in 12.1.

Binomial Coefficients

If j and n are integers with $0 \le j \le n$, the symbol $\binom{n}{j}$ is defined as

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

Calculate the binomial coefficients

$\binom{9}{4}$

$\binom{100}{3}$

This helps up because the values of Pascal's Triangle are in fact binomial coefficients!

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdot \quad \cdot \quad \cdot \quad \binom{n}{n-1} \quad \binom{n}{n}$$

Binomial Theorem

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} a^j$$

Use the Binomial Theorem to expand the following:

$(x+y)^4$
$(2y-3)^4$

The Binomial theorem may be used to find a particular term of a binomial expansion:

Based on the expansion of $(x + a)^n$, the term containing x^j is

$$\binom{n}{n-j} a^{n-j} x^j \qquad \textbf{(3)}$$

Find the find the coefficient of y^8 in the expansion of $(2y + 3)^{10}$

Find the 6th term in the expansion of $(x + 2)^9$