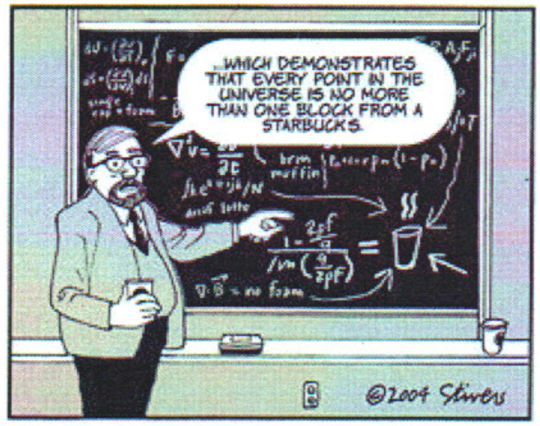
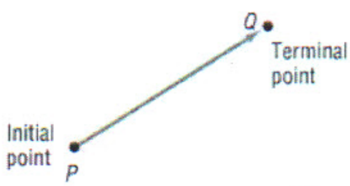
Precalculus
Lesson 9.4: Vectors
Mrs. Snow, Instructor



Many concepts in science involve applications of mathematics that measure certain quantities by their magnitude like length, mass, area, temperature, or energy. Only one number is needed to describe a length of 7 inches or 5°C for example. This single quantity is called **scalar.**

There are, however, many applications that involve not only the *magnitude* of an object but also, the *direction* of the displacement.

vector: a quantity that has both magnitude and direction. For example, the flight pattern of a plane, has both *speed (magnitude)* and *direction* of travel. Velocity, acceleration, and force are described by both magnitude and direction and are known as vectors.



(c) Directed line segment $\overrightarrow{PQ}$

P is the initial point
Q is the terminal point

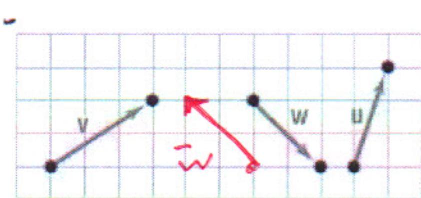
All vectors have two things:

Direction – follow the arrow.

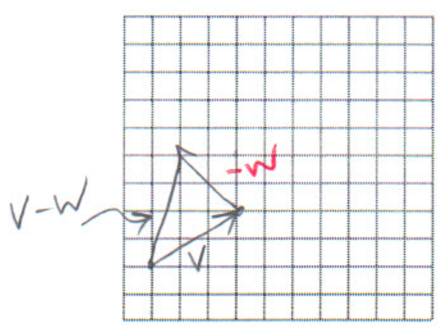
Magnitude – the length of the vector.|| ||

Graphing Vectors

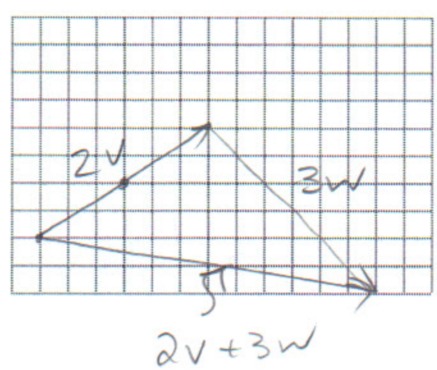
Use the vector to graph each of the following vectors:



$v - w$



$2v + 3w$



The Position Vector

To compute magnitude and direction of a vector, we need an algebraic way to describe the vector. The algebraic vector v is:

$$\boldsymbol{v = \langle a, b \rangle}$$
$$\boldsymbol{v = \langle horizontal\,, vertical \rangle}$$

a and b are real (scalar) numbers and are called the **components** of the vector.

Vector **v**, may be described with initial point P_1 (x_1, y_1) terminal point P_2 (x_2, y_2)

Vector **v**, is equal to the position vector :

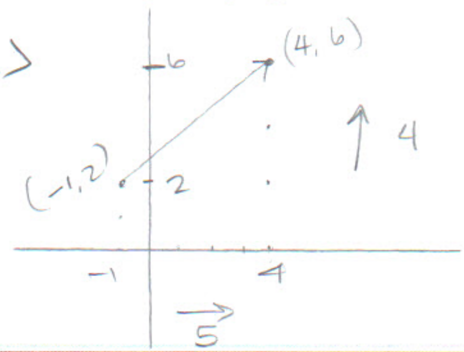
$$\boldsymbol{v = \langle x_2 - x_1, \quad y_2 - y_1 \rangle}$$

Find the position vector **v** with initial point $(-1, 2)$ and terminal point $(4, 6)$.

$V = \langle 4 - (-1), 6 - 2 \rangle$

$\boxed{= \langle 5, 4 \rangle}$

$\langle$horiz, vert$\rangle$



Vectors in terms of *I and j*

A vector of length **1** is called a **unit vector**. Let "**i**" be a unit vector in the x-direction and "**j**" be a unit vector in the y-direction. Any vector in the x-direction can be written as a scalar multiple of **i** and any vector in the y-direction can be written as a scalar multiple of **j.** They are defined as:

$$\boldsymbol{i = \langle 1, 0 \rangle \ and \ j = \langle 0, 1 \rangle}, \ where \| i \| = \sqrt{1^2 + 0^2} \ and \ \| j \| = \sqrt{0^2 + 1^2}.$$
$$v = \langle a, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = \boldsymbol{ai + bj}$$

Algebraic Operations

Vectors may be added, subtracted, or have scalar multiplication. Pretty straight forward, we can treat the numbers as coefficients and i and j as variables. Just remember who they are!

Let $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} = \langle a_2, b_2 \rangle$ be two vectors, and let α be a scalar. Then

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} = \langle a_1 + a_2, b_1 + b_2 \rangle \qquad \textbf{(2)}$$
$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} = \langle a_1 - a_2, b_1 - b_2 \rangle \qquad \textbf{(3)}$$
$$\alpha\mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} = \langle \alpha a_1, \alpha b_1 \rangle \qquad \textbf{(4)}$$
$$\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2} \qquad \textbf{(5)}$$

If $v = 2i + 3j = \langle 2, 3, \rangle$ *and* $w = 3i - 4j = \langle 3, -4 \rangle$,

find: a) $v + w$, b) $v - w$, c) $3v$, d) $2v - 3w$, and e) $\|v\|$

a) $v = 2i + 3j = \langle 2, 3 \rangle$

$+w = 3i - 4j = \langle 3, -4 \rangle$

$5i - j = \langle 5, -1 \rangle$

b) $v = 2i + 3j = \langle 2, 3 \rangle$

$-w = -3i + 4j = -\langle 3, -4 \rangle$

$= -i + 7j = \langle -1, 7 \rangle$

c) $3v = 3(2i + 3j) = 3\langle 2, 3 \rangle$

$= 6i + 9j = \langle 6, 9 \rangle$

d) $2v - 3w = 2(2i + 3j) = 4i + 6j = 2\langle 2, 3 \rangle$

$-3(3i - 4j) = -9i + 12j \quad -3\langle 3, -4 \rangle$

$-5i + 18j \quad \langle -5, 18 \rangle$

e) $\|v\| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

Velocity vector - A vector that represents speed and direction of an object.

Force vector - A vector describing the direction and amount of force acting upon an object

Finding a Vector from its Direction and Magnitude DEGREES!

Given the magnitude $\|v\|$ of a nonzero vector **v** and the **direction angle** α, $0° < \alpha < 360°$, between vectors **v and i**, then:

$$v = \|v\|(\cos\alpha\boldsymbol{i} + \sin\alpha\boldsymbol{j})$$

Writing a Vector When Its Magnitude and Direction Are Given

A ball is thrown with an initial speed of 25 mph in a direction that makes an angle of 30° with the positive x-axis. Express the velocity vector v in terms of I and j. What is the initial speed in the horizontal direction? What is the initial speed in the vertical direction?

$$V = 25\,(\cos 30°\, i + \sin 30°\, j)$$

$$= 25\left(\frac{\sqrt{3}}{2}\, i + \frac{1}{2}\, j\right)$$

$$= \frac{25\sqrt{3}}{2}\, i + \frac{25}{2}\, j$$

(Horiz) (vertical)

$V_{Horizontal} = \frac{25\sqrt{3}}{2} \sim 21.65$ mph $V_{ertical} = 12.5$ mph

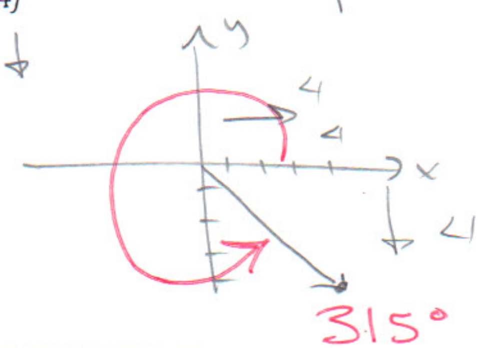
Finding the Direction Angle of a Vector

Draw a picture!

Find the direction angle α for $v = 4i - 4j$

$\boxed{\alpha = 315°}$ $\rightarrow$ $\downarrow$

Counter clockwise!!



Finding the Actual Speed and Direction of an Aircraft

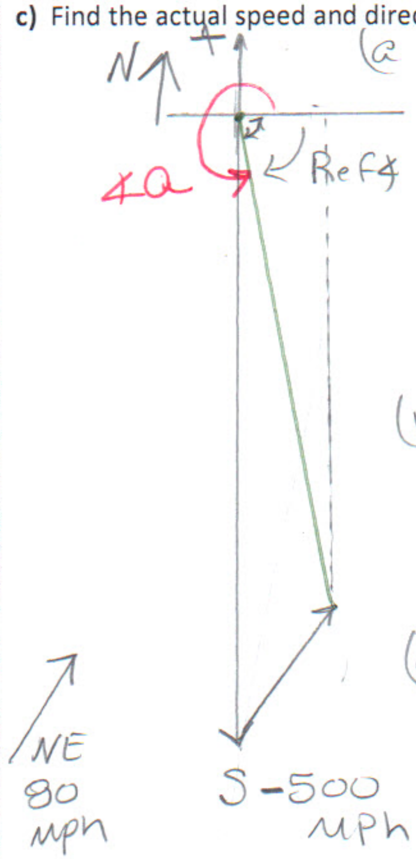
Vertical only

A Boeing 737 aircraft maintains a constant airspeed of 500 mph headed due south. The jet stream is 80 mph in the northeasterly direction. ← implies N45°E

a) Express the velocity v_a of the 737 relative to the air and velocity v_w of the jet stream in terms of I and j.

b) Find the velocity of the 737 relative to the ground. Sum of jet & wind vectors

c) Find the actual speed and direction of the 737 relative to the ground.



(a) $V_a = -500j$ (↓ negative)

$V_W = 80(\cos 45° i + \sin 45° j)$

$= 80\frac{\sqrt{2}}{2} i + 80\frac{\sqrt{2}}{2} j$

$V_W = 40\sqrt{2} i + 40\sqrt{2} j$

(b) $V_a = -500j$

$+ V_W = 40\sqrt{2} i + 40\sqrt{2} j$

$V_g = 40\sqrt{2} i + (40\sqrt{2} - 500) j$

Horiz. Vert

(c) speed $= \|v\| = \sqrt{a^2+b^2}$

$= \sqrt{(40\sqrt{2})^2 + (40\sqrt{2} - 500)^2}$

$\boxed{\approx 447 \text{ mph}}$

∡a

$\tan a = \frac{o}{a} = \frac{40\sqrt{2}-500}{40\sqrt{2}} \sim -82.37°$

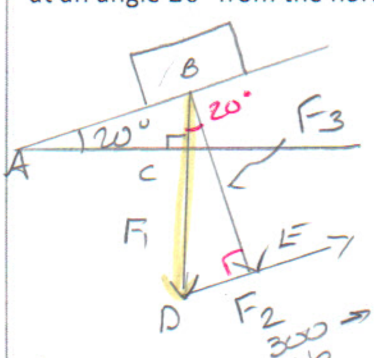
90-82.37 $\sim$ 7.3°

Ref ∡

∴ Direction = S 7.3° E

Finding the Weight of a Piano

Two movers require a magnitude of force of 300 pounds to push a piano up a ramp inclined at an angle 20° from the horizontal. How much does the piano weigh?



3 Forces: F_1 = Force of gravity (weight)

Want F_1 Weight

F_2 = Move piano 300 Lb

F_3 = force piano against ramp

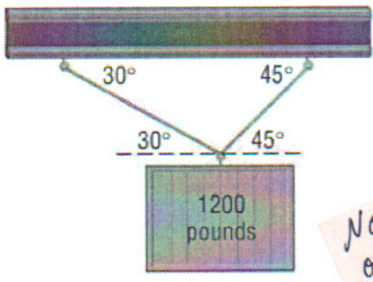
△ABC similar to △BDE

$\sin 20° = \frac{\|F_2\|}{\|F_1\|} = \frac{300}{F_1}$

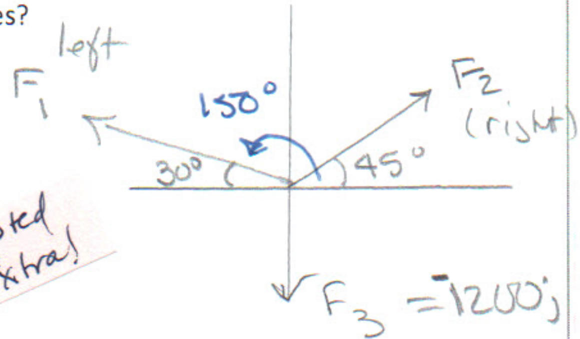
$F_1 = \frac{300}{\sin 20°} \sim \underline{877 lb}$

An Object in Static Equilibrium: the object is at rest and the sum of all forces acting on the object is zero, a.k.a. the resultant force is zero. $F_1 + F_2 + F_3 = 0$ Stable (Static Equilib.)

A box of supplies that weighs 1200 pounds is suspended by two cables attached to the ceiling. What are the tensions in the two cables?



Not to be tested on this = Extra!



$$F_1 = \|F_1\|(\cos 150\, i + \sin 150\, j)$$
$$= \|F_1\|\left(-\frac{\sqrt{3}}{2} i + \frac{1}{2} j\right)$$
$$= -\frac{\sqrt{3}}{2}\|F_1\| i + \frac{1}{2}\|F_1\| j$$

$$F_2 = \|F_2\|(\cos 45\, i + \sin 45\, j)$$
$$= \|F_2\|\left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j\right)$$
$$= \frac{\sqrt{2}}{2}\|F_2\| i + \frac{\sqrt{2}}{2}\|F_2\| j$$

$$F_3 = -1200j$$

Static equilibrium: Horizontal components = 0
Vertical components = 0

Horizontal (i)

$$-\frac{\sqrt{3}}{2}\|F_1\| + \frac{\sqrt{2}}{2}\|F_2\| = 0$$

Vertical (j)

$$\frac{1}{2}\|F_1\| + \frac{\sqrt{2}}{2}\|F_2\| - 1200 = 0$$
$$\frac{1}{2}\|F_1\| + \frac{\sqrt{2}}{2}\|F_2\| = 1200$$

Solve for $\|F_1\|$ & $\|F_2\|$

System of Equations

(-1)
$$+\frac{\sqrt{3}}{2}\|F_1\| - \cancel{\frac{\sqrt{2}}{2}\|F_2\|} = 0$$
$$\frac{1}{2}\|F_1\| + \cancel{\frac{\sqrt{2}}{2}\|F_2\|} = 1200$$
$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\|F_1\| = 1200$$
$$\|F_1\| = \frac{1200}{\left(\frac{\sqrt{3}+1}{2}\right)} \approx 878.5\text{ lb (left side)}$$

$\|F_1\|$
$$-\frac{\sqrt{3}}{2}(878.51) + \frac{\sqrt{2}}{2}\|F_2\| = 0$$
$$\|F_2\| \sim 1075.9\text{ lb}$$
(Right Side)