Calculus
Lesson 5.1: The Natural Logarithmic Function: Differentiation
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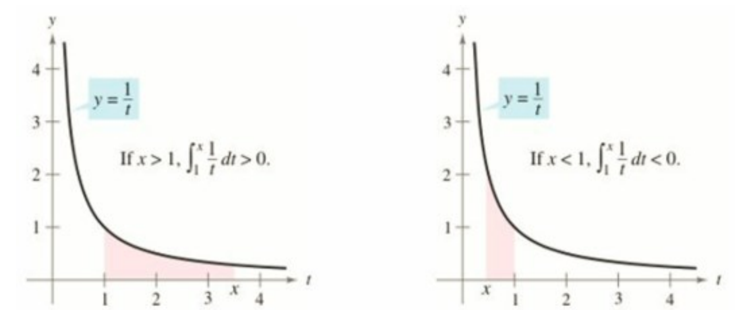
When we were introduced to the General Power Rule, it came with an important disclaimer – it does not apply when $n = -1$. So, what is the antiderivative of $f(x) = \frac{1}{x}$??? Well, that is where the Second Fundamental Theorem of Calculus comes in to play; it will allow us to define this crazy function!

DEFINITION OF THE NATURAL LOGARITHMIC FUNCTION

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.



If $x > 1$, then $\ln x > 0$.

If $0 < x < 1$, then $\ln x < 0$.

THEOREM 5.1 PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

THEOREM 5.2 LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Expand the following logarithms

a. $\ln \frac{10}{9}$

b. $\ln \sqrt{3x + 2}$

c. $\ln \frac{6x}{5}$

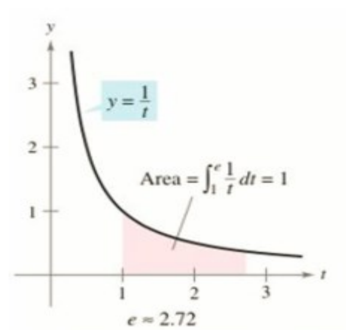
d. $\ln \frac{(x^2 + 3)^2}{x\sqrt[3]{x^2 + 1}}$

The Number e

DEFINITION OF e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$



e is the base for the natural logarithm because $\ln e = 1$.

Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

THEOREM 5.3 DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

Differentiation of Logarithmic Functions

a. $\frac{d}{dx}[\ln(2x)]$

b. $\frac{d}{dx}[\ln(x^2 + 1)]$

c. $\frac{d}{dx}[x \ln x]$

d. $\frac{d}{dx}[(\ln x)^3]$

Logarithmic Properties as Aids to Differentiation

- Differentiate:

$f(x) = \ln\sqrt{x + 1}$

$f(x) = \ln \frac{x(x^2 + 1)^2}{\sqrt{2x^3 - 1}}$.

And yes, we can use logarithms to help us in differentiating nonlogarithmic functions!

Logarithmic Differentiation

Find the derivative of:

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}} \quad , \quad x \neq 2$$

Remember that logarithims are undefined for negative numbers, hence, you will find expressions with the $ln|x|$. That is OK, we are able to differentiate as if the absolute values signs were not present.

THEOREM 5.4 DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Derivative Involving Absolute Value

- Find the derivative of:

$$f(x) = \ln|\cos x|$$

Finding Relative Extrema

- Locate the Relative Extrema of:

$$y = \ln\left(x^2 + 2x + 3\right)$$