Precalculus

Lesson 9.5: The Dot Product

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The concept of the dot product is used in calculus and in the applications of vectors in physics and engineering.

If $v = a_1 i + b_1 j = \langle a_1, b_1 \rangle$ and $w = a_2 i + b_2 j = \langle a_2, b_2 \rangle$ are vectors, then their dot product, denoted by $\boldsymbol{v \cdot w}$, is defined by

$$\boldsymbol{v \cdot w = a_1 a_2 + b_1 b_2}$$

say: "v dot w"

horizontal product + vertical product

Given:

$$v = 2i - 3j \quad and \quad w = 5i + 3j$$

Find the following dot products:

a) $v \cdot w = (2)(5) + (-3)(3) = 10 - 9 = 1$

b) $w \cdot v = 5(2) + 3(-3) = 10 - 9 = 1$

c) $v \cdot v = 2(2) + (-3)(-3) = 4 + 9 = 13$

d) $w \cdot w = 5(5) + 3(3) = 25 + 9 = 34$

e) $\|v\| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

f) $\|w\| = \sqrt{5^2 + 3^2} = \sqrt{25+9} = \sqrt{34}$

The following properties of the Dot Product are useful in solving problems involving the Dot Product:

$$u \cdot v = v \cdot u \quad \text{commutative}$$

$$(au) \cdot v = a(u \cdot v) = u \cdot (av) \quad \text{associative}$$

$$u \cdot (v + w) = u \cdot v + u \cdot w \quad \text{distributive}$$

$$v \cdot v = \|v\|^2$$

$$0 \cdot v = 0$$

The Dot Product Theorem

If we have u and v be vectors with initial points at the origin, the angle θ that is between u and v is $0 < \theta < \pi$.

$$\boldsymbol{u \cdot v = \|u\|\|v\| cos\theta}$$

$$\boldsymbol{cos\theta = \frac{u \cdot v}{\|u\|\|v\|}}$$

Find the angle θ between $u = 4i - 3j$ *and* $v = 2i + 5j$

with vectors, find Dot product & magnitudes then θ.

$u \cdot v =$

$4(2) + (-3)(5) =$

$8 - 15 =$

-7

$\|u\| = \sqrt{16+9}$

$= \sqrt{25} = 5$

$\|v\| = \sqrt{4+25}$

$= \sqrt{29}$

$\cos\theta = \frac{-7}{5\sqrt{29}}$

$= .259...$

$\theta = \cos^{-1}(.259...)$

$\theta \approx 105.1°$

Orthogonal Vectors (a.k.a. perpendicular)

Two vectors v *and* w are orthogonal, a.k.a. perpendicular, if and only if:

$$v \cdot w = 0$$

Determine whether or not the vectors pair are perpendicular

$v = 2i - j$ *and* $w = 3i + 6j$

$(2)(3) + (-1)(6) = 0$ $\perp$

Parallel Vectors

Two vectors v *and* w are parallel if they are "multiples" of each other.

Determine whether the vectors in each pair are parallel. or not

$v = 2i - j$ *and* $w = 6i - 3j$

$3v = 6i - 3j$ ↩ yep! parallel

$w = 3i + 4j$ *and* $r = 5i + 2j$

$3(x) = 5$ $\quad \frac{5}{3}(4) \neq 2$ Nope! not parallel

$x = \frac{5}{3}$

Work

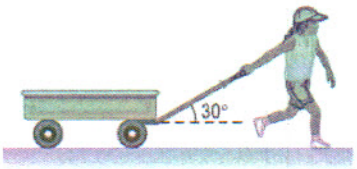
Work equals force times distance:

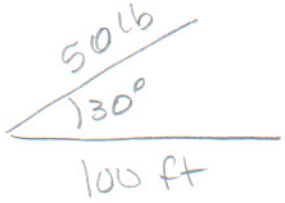
$$W = F \cdot D.$$

English units of force is pounds (lbs.)

When the force acting on the object is at an angle, remember to break it into its horizontal and vertical components.

A girl is pulling a wagon with a force of 50 pounds. How much work is done in moving the wagon 100 feet if the handle makes an angle of 30° with the ground?





$W = F \cdot D$

$F = 50(\cos 30\, i + \sin 30\, j)$

$= 50\left(\frac{\sqrt{3}}{2} i + \frac{1}{2} j\right)$

$F = 25\sqrt{3}\, i + 25 j$

$D = 100\, i + 0 j$

$W = F \cdot D = (100)(25\sqrt{3}) + 25(0)$

$= 2500\sqrt{3} \approx 4330$ Lb.