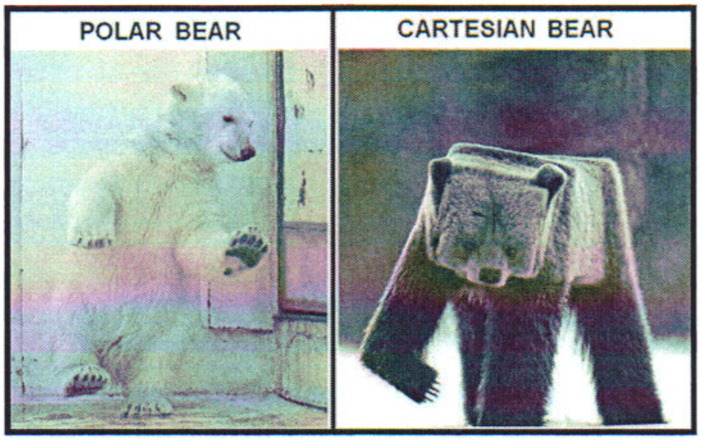
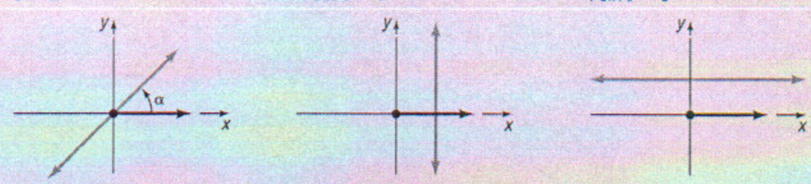
Precalculus
Lesson 9.2 Graphs of Polar Equations
Mrs. Snow, Instructor



As we studied last section points may be described in polar form or rectangular form. Likewise an equation may be written using either polar or rectangular coordinates. Depending on specific equation, one form may be easier to understand and graph than the other. Below are some common polar graphs and their equations written in both polar and rectangular forms.

Lines			
Description	Line passing through the pole making an angle α with the polar axis	Vertical line	Horizontal line
Rectangular equation	$y = (\tan\alpha)x$	$x = a$	$y = b$
Polar equation	$\theta = \alpha$	$r\cos\theta = a$	$r\sin\theta = b$
Typical graph			



To plot points we use polar coordinates and a polar grid.

The points (r, θ) *of*

$radius = 6, \quad with\ \theta = \frac{5\pi}{6}$ *and*

$radius = 4, \quad with\ \theta = \frac{\pi}{4}$

Special graphs:

θ = constant –a line at angle θ

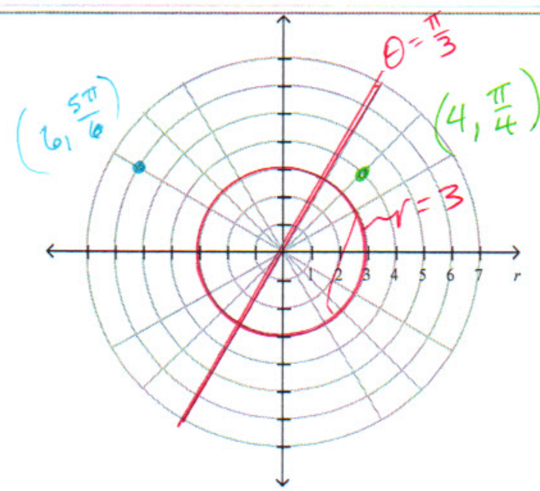
r=constant –a circle of radius r

Sketch the graph of the equation and:
How may we write in rectangular coordinates? – Equation of Circle

$r = 3$

$x^2 + y^2 = r^2$

$x^2 + y^2 = 9$



$\theta = \frac{\pi}{3}$

Graphing a Polar Equation of a Line:

Some equations can easily be expressed in rectangular coordinates. If this is the case then convert to rectangular coordinates.

Identify and graph the equation

$$\theta = \frac{\pi}{4}$$

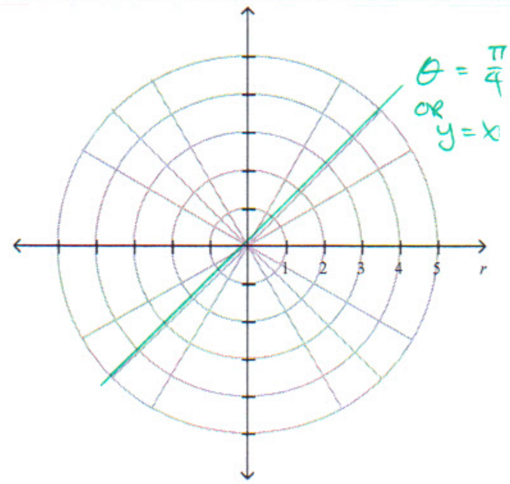
take the tangent of both sides:

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\downarrow$$

$$\frac{y}{x} = 1$$

$$y = x$$



Remember the formulas from section 1 that relate x *and* y *to* r *and* θ:

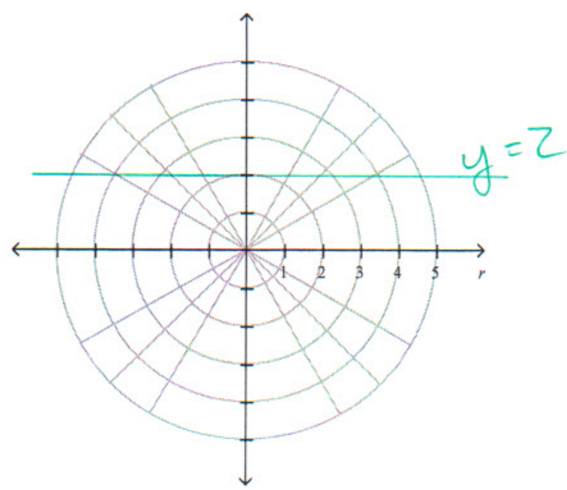
Identify and graph the equation

$$r\sin\theta = 2$$

$r\sin\theta = y$

$$\downarrow$$

$$y = 2$$



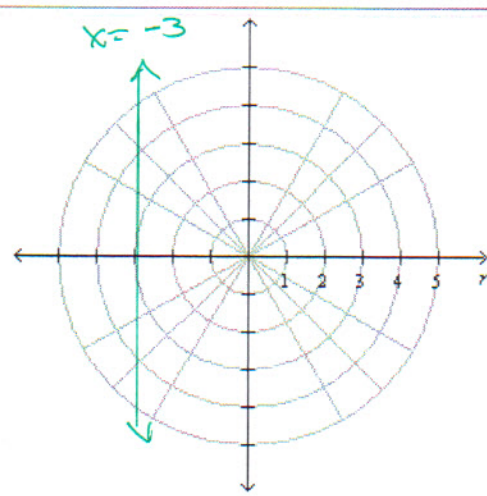
Identify and graph the equation

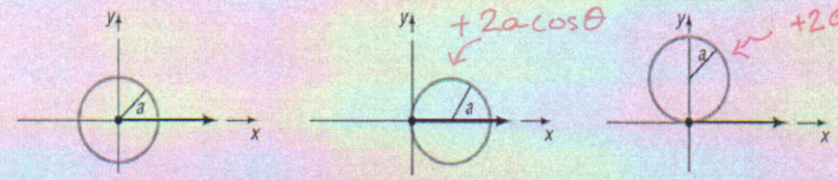
$$r\cos\theta = -3$$

$r\cos\theta = x$

$$\downarrow$$

$$x = -3$$



Circles			where "a" is circle radius
Description	Center at the pole, radius a	Passing through the pole, tangent to the line $\theta = \frac{\pi}{2}$, center on the polar axis, radius a	Passing through the pole, tangent to the polar axis, center on the line $\theta = \frac{\pi}{2}$, radius a
Rectangular equation	$x^2 + y^2 = a^2, \quad a > 0$	$x^2 + y^2 = \pm 2ax, \quad a > 0$	$x^2 + y^2 = \pm 2ay, \quad a > 0$
Polar equation	$r = a, \quad a > 0$	$r = \pm 2a\cos\theta, \quad a > 0$	$r = \pm 2a\sin\theta, \quad a > 0$
Typical graph			

Notice orientation of circle and trig function in the equation. A "+2a" circle will be on the positive axis & ~~a "−2a" circle will be on the negative axis~~

Graphing a Circle

Sketch the polar equation (transform the equation into its rectangular form)

$$r = 4\sin\theta$$

$2a = 4$

$a = 2$ (radius = 2)

need $r\sin\theta$

$r^2 = 4\underbrace{r\sin\theta}$

Multiply each side through by r

$x^2 + y^2 = 4y$

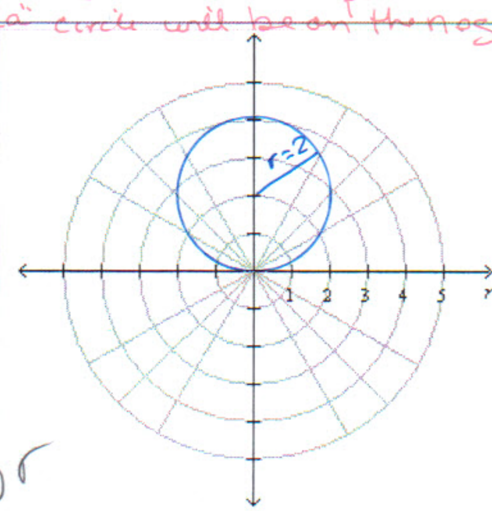
$x^2 + y^2 - 4y + 4 = 0 + 4$

complete the square

$x^2 + (y-2)^2 = 4$

circle with center at $(0, 2)$

$r = 2$!



Sketch the polar equation

$$r = -2\cos\theta$$

$2a = -2$

$a = -1$

(radius = 1)

negative: tells us we are on the negative side

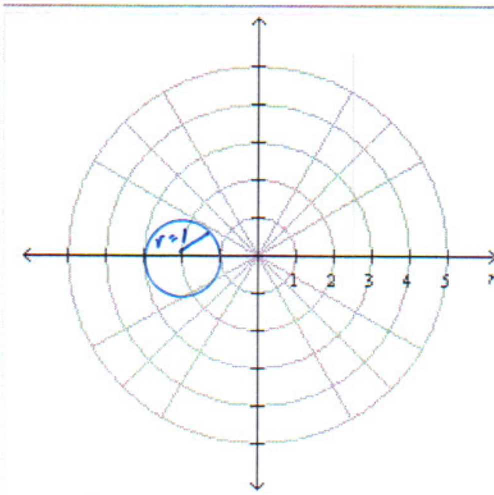
Also:

$r^2 = -2r\cos\theta$

$x^2 + y^2 = -2x$

$x^2 + 2x + 1 + y^2 = 0 + 1$

$(x+1)^2 + y^2 = 1$ — $r = 1$ center $(-1, 0)$



Cardioid – heart shaped (pg. 581)

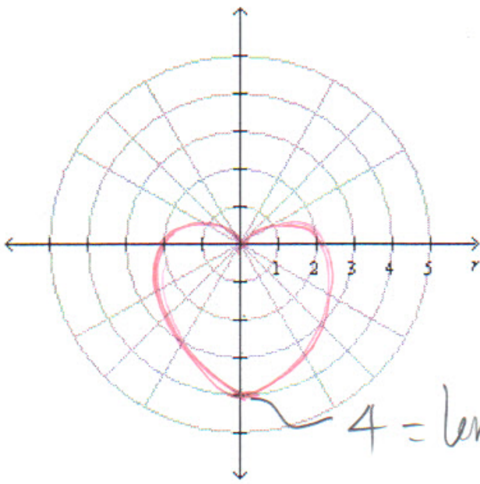
graph $r = 2 - 2\sin\theta$

a= 2 b= 2 → Same

The numbers indicate a shape of Cardioid

equation has sine so along $\pi/2$ axis (vertical)

Negative means: Negative side (bottom) length = $2(2) = 4$



4 = length

Whenever you cannot remember how to graph the polar equation, you can always graph a period of the trig function from $0 \le \theta < 2\pi$ *and transfer the data over to a polar graph. Don't rely on memorizing an equation and associated graph shape, you will want a backup method!!*

You don't need to do this! This is your failsafe!!

Table of values *(use values for theta that yield friendly values for r)*:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	1/2	1	1/2	0	-1	0
$r = 2 - 2\sin\theta$	2	1	0	1	2	4	2

Other Equations (pg. 581)

as "a" increases, limaçon loop decreases — or disappears

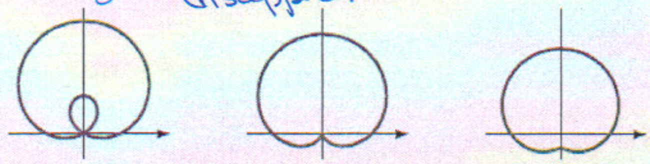
Limaçons

$r = a \pm b \sin\theta$

$r = a \pm b \cos\theta$

$(a > 0, b > 0)$

Orientation depends on the trigonometric function (sine or cosine) and the sign of b.



$a < b$ limaçon with inner loop	$a = b$ cardioid	$a > b$ dimpled limaçon

Name	Limaçon inner loop smaller	Cardioid number & coefficient equal	Limaçon **no** inner loop a has a dimple bigger — smaller
Polar Equation	$r = a \pm b\cos\theta$ $r = a \pm b\sin\theta$ $a < b$	$r = a \pm a\cos\theta$ $r = a \pm a\sin\theta$ $a = b$	$r = a \pm b\cos\theta$ $r = a \pm b\sin\theta$ $a > b$

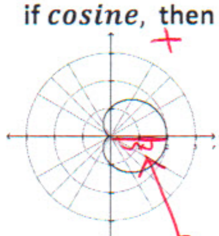
Equations in terms of cosine will be symmetrical about the polar axis (horizontal).
Equations in terms of sine will be symmetrical about the $\frac{\pi}{2}$ axis (vertical).

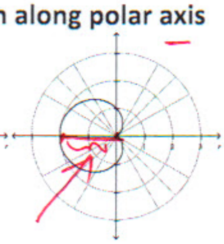
Cardioid graphs

$\underline{a = b, \quad distance\ on\ axis\ is\ 2a}$

if *cosine*, then along polar axis

+ −





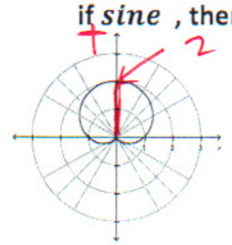
2

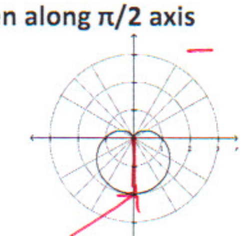
$r = 1 + \cos\theta$ $\qquad$ $r = 1 - \cos\theta$

a = 1
length of cardioid
= 2(1) = 2

if *sine*, then along π/2 axis

+ 2 −





2

$r = 1 + \sin\theta$ $\qquad$ $r = 1 - \sin\theta$

2(1) = 2

Limaçon graphs

to find dimple or loop

$b - a$

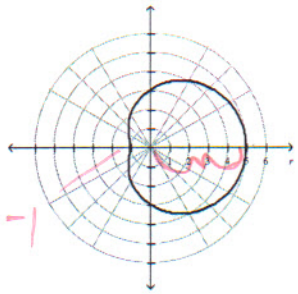
$$r = a \pm \mathbf{b} \cos \boldsymbol{\theta}$$
$$r = a \pm \mathbf{b} \sin \boldsymbol{\theta}$$

if *cosine*: along polar axis
if *sine*: along $\frac{\pi}{2}$ axis

to find the bottom of Limaçon, add

$b + a$

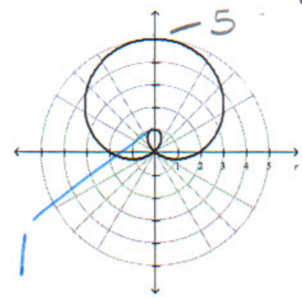
a. Limacon no inner loop if :
$a > b$



$b + a = 5$
$b - a = -1$
$2b = 4$
$b = 2$
$\therefore a = 3$

So: $r = 3 + 2\cos\theta$

b. Limacon has an inner loop if:
$a < b$



$b + a = 5$
$b - a = 1$
$2b = 6$
$b = 3$
$\therefore a = 2$

So: $r = 2 + 3\sin\theta$

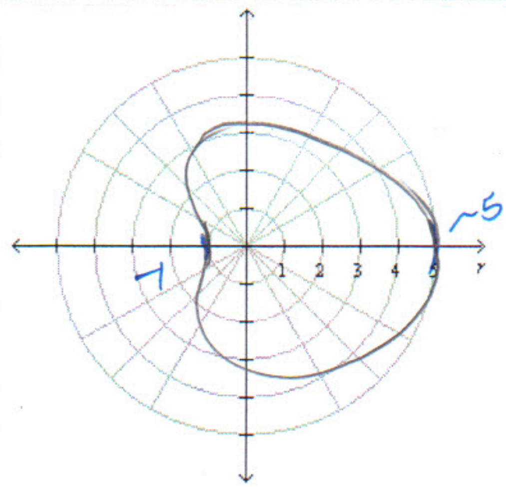
Graphing a limaçon without an inner loop

Sketch the graph of the equation
$r = 3 + 2cos\,\theta$

$a = 3$ bigger
$b = 2$

"a" bigger – No loop.
cosine ⟷

$b + a = 2 + 3 = 5$
$b - a = 2 - 3 = -1$

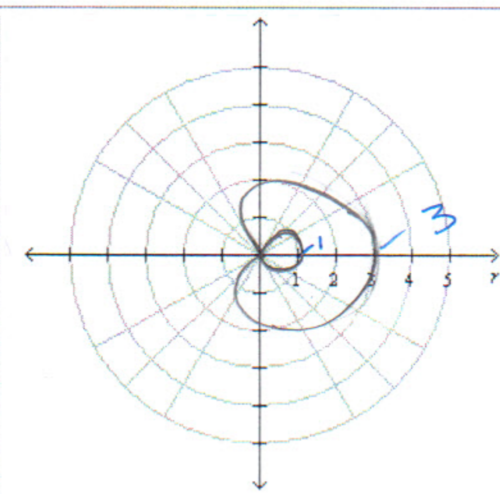


Graphing a limaçon with an inner loop

$r = 1 + 2\ cos\ \theta$

a – smaller loop
cosine ⟷

$b + a = 2 + 1 = 3$
$b - a = 2 - 1 = 1$



More Equations

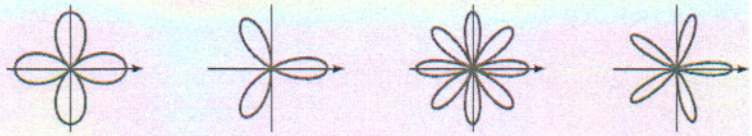
Roses

$r = a \sin n\theta$

$r = a \cos n\theta$

n-leaved if n is odd

$2n$-leaved if n is even



$r = a\cos 2\theta$ 4-leaved rose | $r = a\cos 3\theta$ 3-leaved rose | $r = a\cos 4\theta$ 8-leaved rose | $r = a\cos 5\theta$ 5-leaved rose

Rose with even/odd petals

$$n: \begin{cases} odd = n \ petals \\ even = 2n \ petals \end{cases}$$

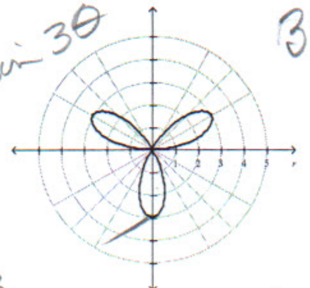
$a = length\ of\ petal$

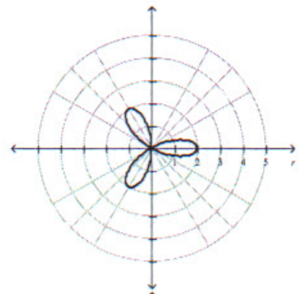
$\boldsymbol{r = a\sin n\theta}$

$\boldsymbol{r = a\cos n\theta}$

$r = 3\sin 3\theta$

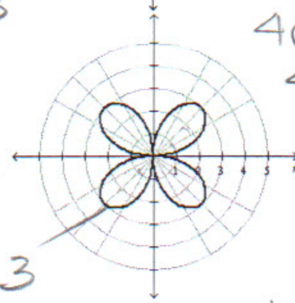
3 petals
$n = 3$

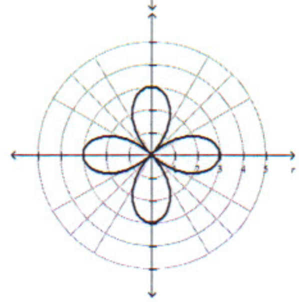




$a = 3$

4 petals
$4 = 2n$
$n = 2$





$a = 3$

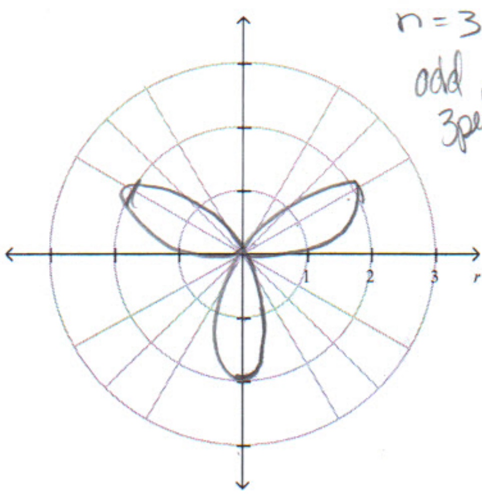
n

$r = 3\sin 2\theta$

Graphing a Polar Equation: n-leaved rose (petals)

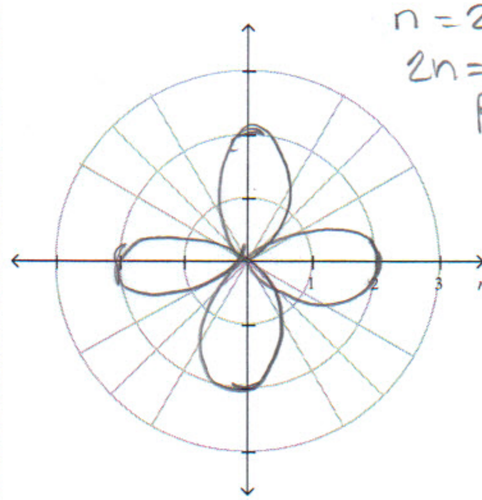
$r = 2\sin 3\theta$

$a = 2$
$n = 3$
odd
3 petals



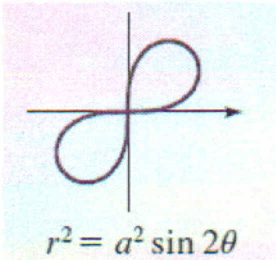
$r = 2\cos 2\theta$

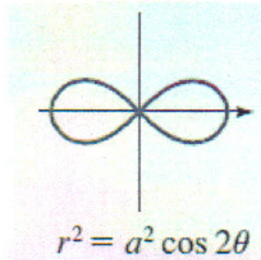
$a = 2$
$n = 2$ even
$2n = 4$ petals



Lemniscates – Figure 8 shaped curves

$r^2 = a^2 sin2\theta$ $\quad$ $r^2 = a^2 cos2\,\theta$ $\quad$ $a = petal\ length$

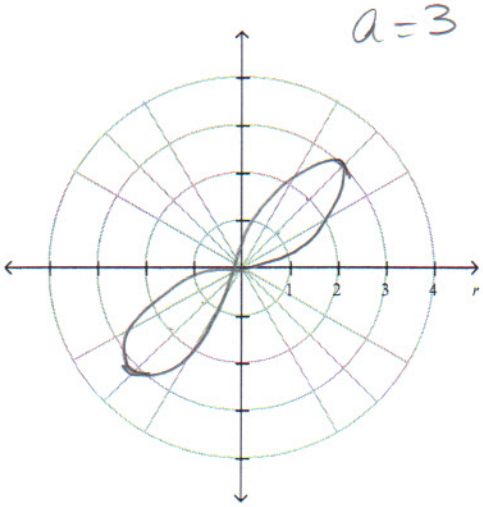




$r^2 = 9sin2\theta$

$a^2 = 9$

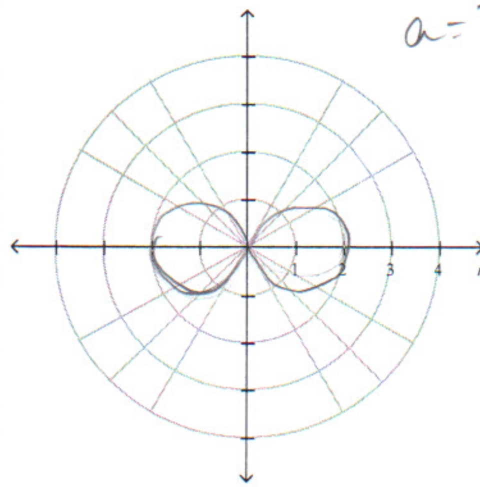
$a = 3$



$r^2 = 2^2cos2\theta$

$2^2 = a^2$

$a = 2$



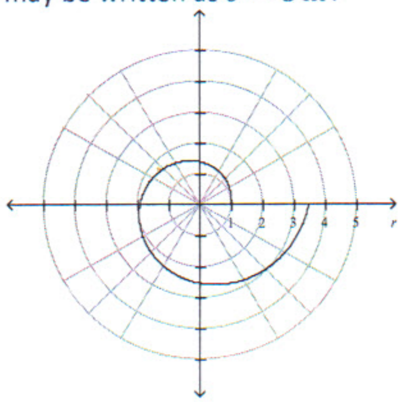
Graphing a Polar Equation (spiral)

It is the locus of points corresponding to the locations over time of a point moving away from a fixed point with a constant speed along a line which rotates with constant angular velocity.

There are several equations that will produce a spiral. The **logarithmic spiral**

$$r = e^{\theta/5}$$

may be written as $\theta = 5\ln r$



Archimedes Spiral is in the form of

$$r = a\theta$$