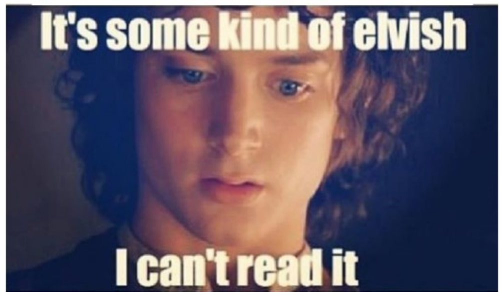
Calculus
Lesson 4.5: Integration by Substitution
Mrs. Snow, Instructor

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$



The role of substitution in integration is comparable to the role of the Chain Rule in differentiation. In this section you will study techniques for integrating **composite functions**.

The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a **"*u*-substitution."**

- With pattern recognition you will perform the substitution mentally.
- With change of variables you will write the substitution steps.

THEOREM 4.13 ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Note that the composite function in the integrand has an *outside function f* and an *inside function g*.
Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

$$\int \overset{\text{Outside function}}{f}(\underset{\text{Inside function}}{g(x)})\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

Find these integrals.

→ Recognizing the $f\big(g(x)\big)g'(x)$ pattern:

Find:

$$\int (x^2+1)^2(2x)dx$$

$$\int 5cos5x\ dx$$

You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\ dx = k\int f(x)\ dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple.

In such cases, you can multiply and divide by the necessary constant multiple

Find:

$$\int x(x^2 + 1)^2\ dx.$$

With a formal change of variables, you completely rewrite the integral in terms of *u* and *du* (or any other convenient variable).

The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du = F(u) + C.$$

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\, dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Using change of variables find:

$$\int \sqrt{2x - 1}\, dx.$$

$$\int x\sqrt{2x-1}\, dx$$

$$\int sin^2 3x\ cos3x\ dx$$

$$\int 2\ sec^2 x\ (tanx + 3)dx$$

One of the most common *u*-substitutions involves quantities in the integrand that are raised to a power.
Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration:**

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Substitution and the General Power Rule

a. $\displaystyle\int 3(3x-1)^4\,dx = \int \underbrace{(3x-1)^4}_{u^4}\underbrace{(3)\,dx}_{du}$

b. $\displaystyle\int (2x+1)(x^2+x)\,dx = \int \underbrace{(x^2+x)^1}_{u^1}\underbrace{(2x+1)\,dx}_{du}$

c. $\displaystyle\int 3x^2\sqrt{x^3-2}\,dx = \int \underbrace{(x^3-2)^{1/2}}_{u^{1/2}}\underbrace{(3x^2)\,dx}_{du}$

d. $\displaystyle\int \frac{-4x}{(1-2x^2)^2}\,dx =$

e. $\displaystyle\int \cos^2 x \sin x\,dx =$

When using *u*-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable *u* rather than to convert the antiderivative back to the variable *x* and evaluate at the original limits.
This change of variables is stated explicitly in the following theorem.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Change of Variables

Evaluate $\int_0^1 x(x^2+1)^3\,dx$.

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$.

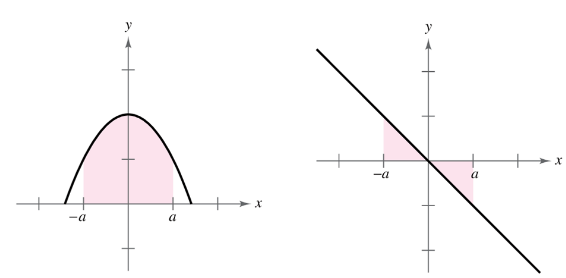
Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the *y*-axis or about the origin by recognizing the integrand to be an even or odd function

THEOREM 4.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0$.



Even function   Odd function

Integration of an Odd Function

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^3 x\cos x + \sin x\cos x\right)dx$$