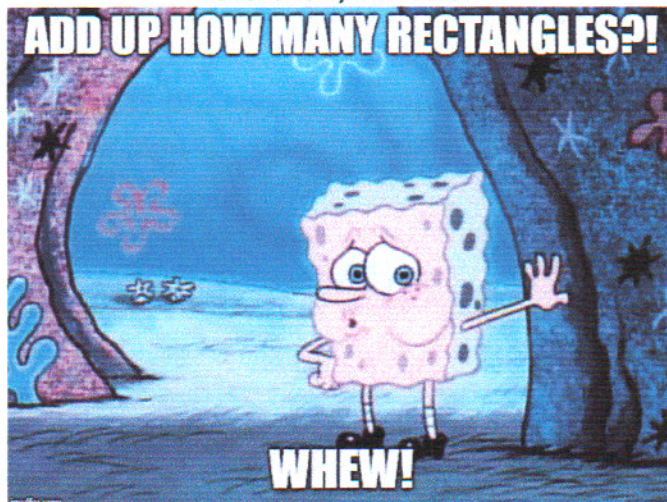


Calculus

Lesson 4.3: Riemann Sums and Definite Integrals

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No fears, we are finished with summing up rectangular areas. However, we should understand the relationship between the definite integral and approximating the area of a region by using rectangles:

DEFINITION OF DEFINITE INTEGRAL

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

(Height) (width) = area !!

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx. \quad \star$$

The limit is called the **definite integral** of f from a to b . The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM 4.4 CONTINUITY IMPLIES INTEGRABILITY

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x) dx$ exists.

THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x) dx.$$

Definite Integrals

$$\int_1^3 4 dx = 4x \Big|_1^3 = 4(3) - 4(1) = 12 - 4 = \underline{\underline{8}}$$

$$\int_0^3 (x+2) dx = \frac{x^2}{2} + 2x \Big|_0^3 = \frac{3^2}{2} + 2(3) - \left(\frac{0}{2} + 2(0)\right) = \frac{9}{2} + 6 = \underline{\underline{\frac{21}{2}}}$$

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then we define $\int_a^a f(x) dx = 0$.
2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x) dx = -\int_a^b f(x) dx$.

THEOREM 4.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a , b , and c , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$
2. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$

Using Additive Interval Property

$$\begin{aligned}\int_{-1}^1 |x| dx &= \int_{-1}^0 -x dx + \int_0^1 x dx \\ &= -\frac{x^2}{2} \Big|_{-1}^0 + \frac{x^2}{2} \Big|_0^1 \\ &= 0 - \left(-\frac{(-1)^2}{2}\right) + \left(\frac{1^2}{2} - \frac{0}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2} \\ &= \underline{\underline{1}}\end{aligned}$$

Evaluation of a Definite Integral

$$\begin{aligned}\int_1^3 (-x^2 + 4x - 3) dx &= \frac{-x^3}{3} + 2x^2 - 3x \Big|_1^3 \\ &= \frac{-27}{3} + 18 - 9 - \left(\frac{-1}{3} + 2 - 3\right) \\ &= \frac{1}{3} - \frac{6}{3} + \frac{9}{3} = \underline{\underline{\frac{4}{3}}}\end{aligned}$$

(OR)

$$\begin{aligned}\int_1^3 -x^2 dx + 4 \int_1^3 x dx + \int_1^3 -x dx &= \frac{-x^3}{3} \Big|_1^3 + 4 \left(\frac{x^2}{2}\right) \Big|_1^3 - \frac{x^2}{2} \Big|_1^3 \\ &= \frac{-27}{3} + \frac{1}{3} + 4 \left(\frac{9}{2} - \frac{1}{2}\right) - 9 + 3 \\ &= \frac{-26}{3} + 16 - 6 = \underline{\underline{\frac{4}{3}}}\end{aligned}$$

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. Provided you can find an antiderivative of f , you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\begin{aligned}\int_a^b f(x) dx &= F(x) \Big|_a^b \\ &= F(b) - F(a)\end{aligned}$$

For instance, to evaluate $\int_1^3 x^3 dx$, you can write

$$\int_1^3 x^3 dx = \left. \frac{x^4}{4} \right|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned}\int_a^b f(x) dx &= \left[F(x) + C \right]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a).\end{aligned}$$

Evaluate each definite integral.

$$\begin{aligned}\int_1^3 (x^2 - 3) dx &= \left. \frac{x^3}{3} - 3x \right|_1^3 = \frac{27}{3} - 9 - \left(\frac{1}{3} - 3 \right) \\ &= \frac{1}{3} + 3 = \frac{1}{3} + \frac{9}{3} = \underline{\underline{\frac{10}{3}}}\end{aligned}$$

$$\begin{aligned}\int_1^4 3\sqrt{x} dx &= \int_1^4 3x^{\frac{1}{2}} dx = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2 \cdot 3x^{\frac{3}{2}}}{2} = 2x^{\frac{3}{2}} \Big|_1^4 \\ &= 16 - 2 = \underline{\underline{14}}\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/4} \sec^2 x dx &= \tan x \Big|_0^{\pi/4} = \tan \frac{\pi}{4} - \tan 0 \\ &= 1 - 0 = \underline{\underline{1}}\end{aligned}$$

A definite integral involving absolute value

$$\int_0^2 |2x-1| dx =$$

$$|2x-1| = \begin{cases} -(2x-1) & x < \frac{1}{2} \\ (2x-1) & x \geq \frac{1}{2} \end{cases}$$

$$\int_0^{\frac{1}{2}} -(2x-1) dx + \int_{\frac{1}{2}}^2 (2x-1) dx$$

$$-x^2 + x \Big|_0^{\frac{1}{2}} + x^2 - x \Big|_{\frac{1}{2}}^2 = -\frac{1}{4} + \frac{1}{2} + 4 - 2 - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + 4 - 2 - \frac{1}{4} + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2}$$

Using the fundamental theorem to find area

Find the area of the region bounded by the graph $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x=0$ and $x=2$ *$y > 0$ on interval $[0, 2]$*

$$\int_0^2 (2x^2 - 3x + 2) dx = \frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x \Big|_0^2$$

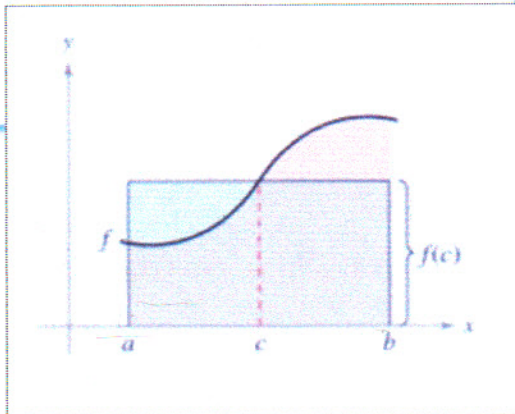
$$\frac{16}{3} - \frac{3}{2}(4) + 4 - 0 = \frac{16}{3} - 6 + 4$$

$$= \frac{16}{3} - \frac{18}{3} + \frac{12}{3} = \frac{10}{3}$$

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x) dx = f(c)(b - a).$$



Using Mean Value Theorem

Find the values of c guaranteed by the mean value theorem for integrals over the given interval. $f(x) = \frac{9}{x^3}$, $[1, 3]$ $\frac{9}{x^3} = 9x^{-3}$

$$\int_1^3 9x^{-3} dx = f(c)(3-1)$$

$$-\frac{9x^{-2}}{2} \Big|_1^3 = f(c)(2)$$

$$-\frac{9}{2(9)} - \left(-\frac{9}{2(1)}\right) = -\frac{1}{2} + \frac{9}{2} = \frac{8}{2} = 4$$

$$4 = 2 f(c)$$

$$\frac{4}{2} = 2 = f(c)$$

$$2 = \frac{9}{c^3}$$

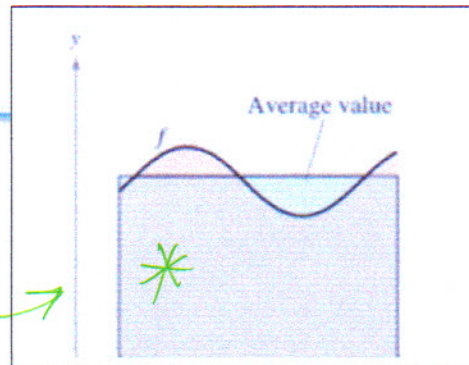
$$\sqrt[3]{c^3} = \sqrt[3]{\frac{9}{2}} \rightarrow c = 1.65$$

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a} \int_a^b f(x) dx.$$



Area under curve is * equal to the rectangle whose height is the average value

Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

$$\begin{aligned} \frac{1}{4-1} \int_1^4 (3x^2 - 2x) dx &= \\ \frac{1}{3} \left[x^3 - x^2 \right]_1^4 &= \\ \frac{1}{3} [(64 - 16) - (1 - 1)] &= \\ \frac{1}{3} (48) &= \boxed{16} \end{aligned}$$

The Definite Integral as a Function

Evaluate the function

$$F(x) = \int_0^x \cos t \, dt = \sin t \Big|_0^x = \sin x - \sin 0 = \boxed{\sin x}$$

at

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \text{ and } \frac{\pi}{2}$$

$$\sin(0) = 0$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

THEOREM 4.11 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on an open interval I containing a , then, for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

← upper bound is variable x
↑ lower bound is a constant

Using the Second Fundamental Theorem of Calculus

$$\frac{d}{dx} \left[\int_0^x \sqrt{t^2 + 1} dt \right] = \sqrt{x^2 + 1}$$

Find the derivative of

$$\frac{d}{dx} \int_{\pi/2}^{x^3} \cos t dt$$

Here upper bound is NOT x but x^3

#1 To evaluate let's define $F(x) = \int_{\pi/2}^x \cos t dt$

$$\text{so: } F'(x) = \cos x$$

#2 We have $F(x^3)$

$$\begin{aligned} \frac{d}{dx} F(x^3) &= F'(x^3)(3x^2) \\ &= \cos x^3 (3x^2) \\ &= 3x^2 (\cos x^3) \end{aligned}$$

Chain Rule

What is $F'(x^3)$?

Use Substitution

$$F'(x) = \cos x$$

$$F'(x^3) = \cos x^3$$

or: x^3

$$\frac{d}{dx} \int_{\pi/2}^{x^3} \cos t dt = \underline{\underline{3x^2 (\cos x^3)}}$$