Calculus
Lesson 2.6: Related Rates
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BRACE YOURSELVES

RELATED RATES ARE COMING

You have now seen how the Chair rule may be used to find dy/dx implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to time.

Two Rates that are Related

Suppose x and y are both differentiable functions of t and are related by the given equation. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

$$y = x^2 + 3$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt} \qquad \frac{dx}{dt} = 2 \quad x = 1$$

$$\frac{dy}{dt} = 2(1)(2)$$

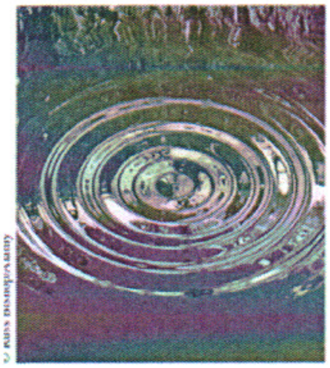
$$\frac{dy}{dt} = 4$$

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined.* Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t.*
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Ripples in a Pond

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the area A of the disturbed water changing?



Total area increases as the outer radius increases.

r = radius $\quad A = \pi r^2 \quad \frac{ft}{sec}$

A = area

$\frac{dr}{dt} = 1 \frac{ft}{sec}$, find $\frac{dA}{dt}$ at $r = 4$

$$\frac{d}{dt} A = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

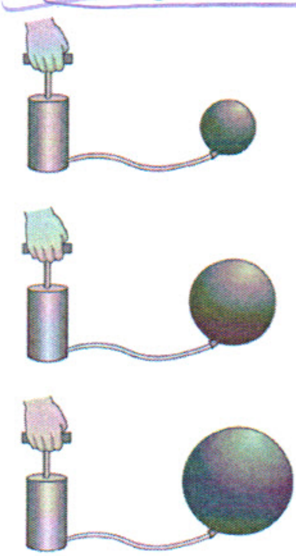
$$\frac{dA}{dt} = (\pi)(2)(4)(1)$$

$$\frac{dA}{dt} = \boxed{8\pi \ ft^2/sec}$$

An Inflating Balloon

VOLUME

Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.



Inflating a balloon

$V = \frac{4}{3}\pi r^3 \quad \frac{dV}{dt} = 4.5 \ ft^3/min$

$\frac{dr}{dt}$ at $r = 2ft$

$$\frac{d}{dt} V = \frac{d}{dt} \frac{4}{3}\pi r^3$$

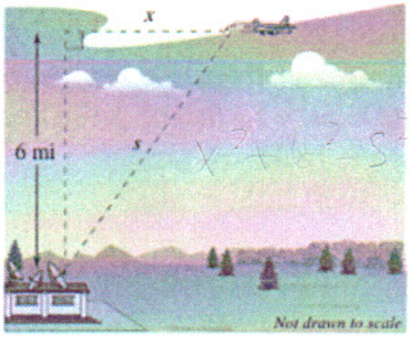
$$\frac{dV}{dt} = \frac{4}{3}\pi \, 3r^2 \frac{dr}{dt}$$

$$4.5 = \frac{4}{3}(\pi)(3)(2^2)\frac{dr}{dt}$$

$$\left(\frac{3}{4}\right)\left(\frac{1}{\pi}\right)\left(\frac{1}{12}\right)(4.5) = \frac{dr}{dt} = .09 \ \frac{ft}{min}$$

The Speed of an Airplane Tracked by Radar

An airplane is flying on a flight path that will take it directly over a radar tracking station. If s is decreasing at a rate of 400 miles per hour when s=10 miles, what is the speed of the plane?



An airplane is flying at an altitude of 6 miles, s miles from the station.

Rt Δ $x^2+6^2=s^2$

at $s=10$ $x^2+6^2=10^2$

$x^2=100-36=64$

$x=8$

$\frac{ds}{dt}=-400$ at $s=10$

So: find $\frac{dx}{dt}$ at $s=10$ $x=8$

$x^2+6^2=s^2$

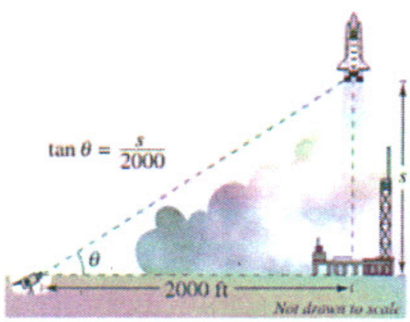
$2x\frac{dx}{dx}=2s\frac{ds}{dt}$

$\frac{dx}{dt}=\frac{\cancel{2}s}{\cancel{2}x}\frac{ds}{dt}=\left(\frac{10}{8}\right)(-400)$

$\boxed{=-500 \text{ mph}}$

A Changing Angle of Elevation

A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the position equation $s = 50t^2$, If the camera is 2000 feet from the launch pad, find the rate of change in the angle of elevation of the camera shown below at 10 seconds after flit-off.



find $\frac{d\theta}{dt}$ at $t=10$ sec

position equ : $s=50t^2$

at $t=10$ $\frac{ds}{dt}=100t$

$s=50(10^2)$

$s=5000$ ft

Rt Δ $\tan\theta=\frac{s}{2000}$

Pythagorean Thm

$2000^2+s^2=h^2$

$\sqrt{2000^2+s^2}=h$

$\cos\theta=\frac{A}{H}=\frac{2000}{\sqrt{2000^2+s^2}}$

$\frac{d}{dt}(\tan\theta)=\frac{d}{dt}\frac{s}{2000}$

$\sec^2\theta\frac{d\theta}{dt}=\frac{1}{2000}\frac{ds}{dt}$

$\frac{d\theta}{dt}=\cos^2\theta\left(\frac{1}{2000}\right)(100t)$

power to power

$\left(\frac{2000}{\sqrt{2000^2+5000^2}}\right)^2\left(\frac{100(10)}{2000}\right)$

$=\frac{2000^{\cancel{2}}}{2000^2+5000^2}\left(\frac{1000}{\cancel{2000}}\right)$

$.069\frac{\text{rad}}{\text{sec}}\left(\frac{180^\circ}{\pi}\right)=4^\circ/\text{sec}$ $\boxed{.069 \text{ rad/sec}}$

Derivatives as Rates of Change

Let's take a quick look at velocity and acceleration. We know that the first derivative may be used to describe velocity. We can also use the second derivative to describe the behavior of velocity. For example, acceleration can tell us if the velocity is increasing or decreasing over time.

Recap

1. $v(t) = ds/dt$ is the instantaneous velocity where $v(t) = s'(t)$
   Velocity is the derivative to the position function. This should make sense as a derivative is a rate of change and the change in position over change in time is velocity.
2. $a(t) = dv/dt = v'(t)$ is acceleration.
   Recall from previous math or physics courses that the change in velocity over change in time is acceleration. Acceleration tells us if an object is increasing its speed, decreasing its speed, or staying constant. For example, a car traveling at exactly 50 mph (never changing its speed) has an acceleration of 0.

Example

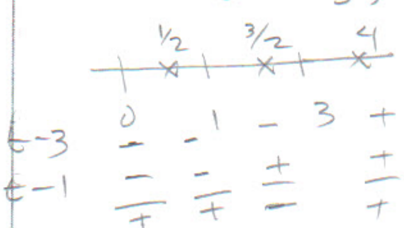
The position of a particle is given by $s(t) = t^3 - 6t^2 + 9t$ where s is measured in meters and t measured in seconds.

a. find the velocity v
b. find the acceleration a of P at time t
c. When is P moving to the right and when to the left Rt vel +, Left vel −
d. find the positions and accelerations of P at the times when it is instantaneously at rest.
e. indicate the motion of P in a diagram

a) $v(t) = 3t^2 - 12t + 9$

b) $a(t) = 6t - 12$

c) criticalpts:
$3t^2 - 12t + 9 = 0$
$\cancel{3}(t^2 - 4t + 3) = 0$
$(t-3)(t-1) = 0$
$t = 3, 1$



d) $t = 0$ $s = 0$
$a = -12\ m/sec^2$

$t = 1$ $s = 4\ m$
$a = -6\ m/sec^2$

$t = 3$ $s = 0\ m$
$a = 6\ m/sec^2$

e)