Calculus
Lesson 2.5: Implicit Differentiation
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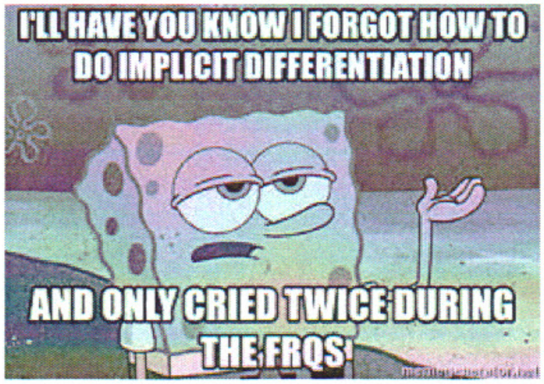
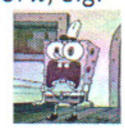


Implicit vs. Explicit: What's the difference? Up to this point, most functions we have dealt with are expressed in **explicit form,** that is the variable y is **explicitly** written as a function of x, e.g. $y = \frac{1}{x}$. Some functions, however, are **implied** by a given equation. An implicit form of the previous example is: $xy = 1$ Note, the implicit form is where the dependent variable is not isolated on one side of the equation. So, no problem we simply solve the equation for y and then take the derivative.



Woops! The problem is that not all equations that we work with can be solved explicitly for y as a function of x. For example, how do we find $\frac{dy}{dx}$ for the equation: $x^2 - 2y^3 + 4y = 2$????

Soooooo, to understand how to find dy/dx implicitly, you must realize that the differentiation is taking place with respect to x. This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the Chain Rule, because you are assuming that y is defined implicitly as a differentiable function of x. → $\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$

Differentiate with respect to x: $\frac{d}{dx}x^3 = 3x^2$ Variables match	$\frac{d}{dx}[y^3] = 3y^2\frac{dy}{dx}$ Variables don't match: use power rule and chain rule
$\frac{d}{dx}[x+3y] = \frac{d}{dx}x + \frac{d}{dx}3y =$ $= 1 + 3\frac{dy}{dx}$	$\frac{d}{dx}[xy^2] = (1)(y^2) + x(2y)\frac{dy}{dx}$ (product rule) $= y^2 + 2xy\frac{dy}{dx}$

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x.*
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Implicit Differentiation: Find $\frac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$

$$\frac{d}{dx}\left[y^3+y^2-5y-x^2\right] = \frac{d}{dx}(-4)$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

factor out $\frac{dy}{dx}$

$$(3y^2+2y-5)\frac{dy}{dx} = -2x$$

solve for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{3y^2+2y-5}}$$

Finding the slope of a graph implicitly

Determine the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$.

$$2x + 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{8y}$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$\frac{\text{change of } y}{\text{Change of } x} \to \text{Slope}$$

@ $\left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{4\left(\frac{1}{-\sqrt{2}}\right)} = \frac{-\sqrt{2}}{-\frac{4}{\sqrt{2}}}$$

$$= \sqrt{2}\left(\frac{\sqrt{2}}{4}\right) = \frac{2}{4}$$

$$\boxed{= \frac{1}{2}}$$

Determine the slope of the graph of $3(x^2+y^2)^2 = 100xy$ at the point (3,1).

$$(3)(2)(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 100y + 100x\frac{dy}{dx}$$

$$6(x^2+y^2)(2)\left(x+y\frac{dy}{dx}\right) =$$

$$12(x^2+y^2)\left(x+y\frac{dy}{dx}\right) =$$

distribution

$$12x(x^2+y^2) + 12(x^2+y^2)\left(y\frac{dy}{dx}\right) = 100y + 100x\frac{dy}{dx}$$

$$12y(x^2+y^2)\frac{dy}{dx} - 100x\frac{dy}{dx} = 100y - 12x(x^2+y^2)$$

collect $\frac{dy}{dx}$ terms

$$\left(12y(x^2+y^2) - 100x\right)\frac{dy}{dx} = 100y - 12x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{100y - 12x(x^2+y^2)}{-100x + 12y(x^2+y^2)} = \frac{\cancel{4}(25y - 3x(x^2+y^2))}{\cancel{4}(-25x+3y(x^2+y^2))}$$

Evaluate at (3,1)

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(9+1)}{-25(3)+3(1)(9+1)} = \frac{25-9(10)}{-25+3(10)}$$

$$= \frac{25-90}{-75+30} = \frac{-65}{-45} = \boxed{\frac{13}{9} = \text{slope}}$$

Find the second derivative Implicitly

Given $x^2+y^2=25$, find $\frac{d^2y}{dx^2}$

* Substitute

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Substitute

$$\frac{d^2y}{dx^2} = \frac{-1(y) - (-x)\frac{dy}{dx}}{y^2}$$

$$= \frac{-y + x\left(\frac{-x}{y}\right)}{y^2}$$

$$\frac{\frac{(y)}{(y)}y - \frac{x^2}{y}}{y^2}$$

*

$$\frac{\frac{-y^2-x^2}{y}}{y^2} = \frac{-1(y^2+x^2)}{y^3} = \boxed{\frac{-25}{y^3}}$$

Finding a tangent line to a graph

Find the tangent line to the graph given by $x^2(x^2+y^2) = y^2$ *at* $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$= x^4 + x^2y^2 - y^2 = 0$$

$$4x^3 + 2xy^2 + 2x^2y\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$2x^2y\frac{dy}{dx} - 2y\frac{dy}{dx} = -4x^3 - 2xy^2$$

factor

$$2y(x^2-1)\frac{dy}{dx} = -2x(2x^2+y^2)$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x(2x^2+y^2)}{\cancel{2}y(x^2-1)}$$

$$\frac{dy}{dx} = \frac{-x(2x^2+y^2)}{y(x^2-1)} \text{ at } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$= \frac{-\cancel{\left(\frac{\sqrt{2}}{2}\right)}\left(2\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2\right)}{\cancel{\frac{\sqrt{2}}{2}}\left(\left(\frac{\sqrt{2}}{2}\right)^2 - 1\right)}$$

$$= \frac{(-1)\left(2\left(\frac{2}{4}\right) + \frac{2}{4}\right)}{\frac{2}{4} - 1}$$

$$= \frac{(-1)\left(1 + \frac{1}{2}\right)}{\left(\frac{1}{2} - 1\right)} = \frac{(+1)\left(\frac{3}{2}\right)}{\left(+\frac{1}{2}\right)} = \frac{3}{\cancel{2}}\left(\frac{\cancel{2}}{1}\right)$$

$$= \boxed{3 = m}$$

Equation:

$$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$$

$$y = 3x - \frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= 3x - \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \boxed{3x - \sqrt{2} = y}$$