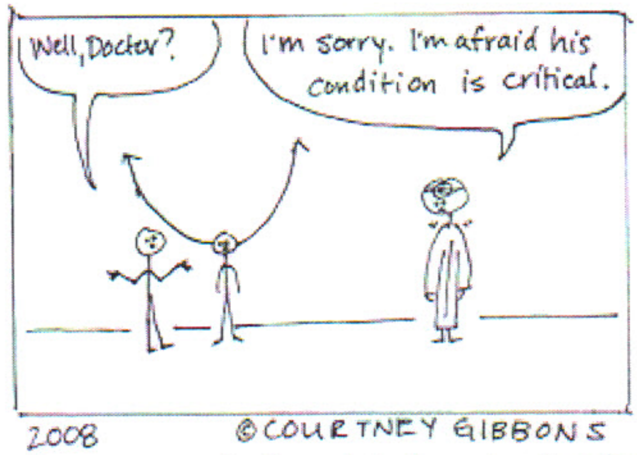
Calculus
Lesson 3.1: Extrema on an Interval
3.2: Rolle's Theorem and the Mean Value Theorem
Mrs. Snow, Instructor



Because calculus is used in so many applications of physics and engineering, finance, and so on, there is a lot of effort devoted to studying the behavior of a function over an interval. Where is the minimum value? Is there a minimum value? Where is the function increasing or decreasing? Is the object starting to slow down? These are just a few of the questions that may be answered by understanding what the extrema are on a given interval.

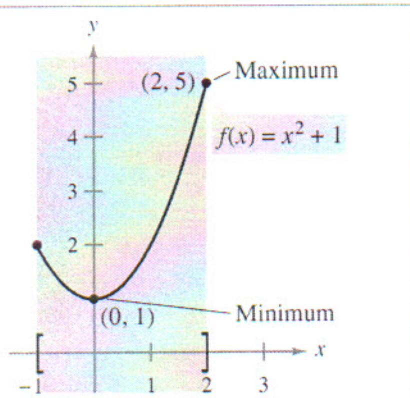
Definition of Extrema

DEFINITION OF EXTREMA

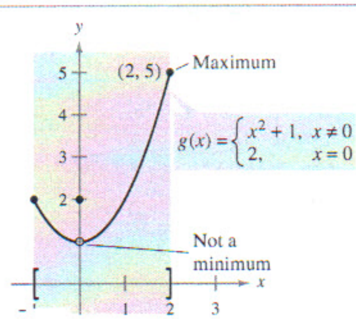
Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \ge f(x)$ for all x in I.

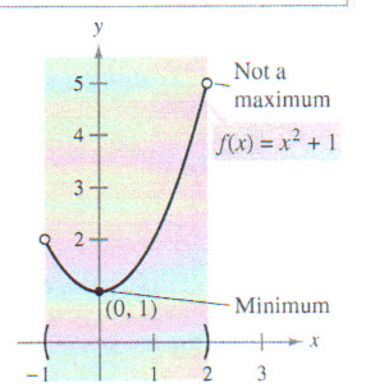
The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval.



(a) f is continuous, $[-1, 2]$ is closed.



(c) g is not continuous, $[-1, 2]$ is closed.
Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**



(b) f is continuous, $(-1, 2)$ is open.

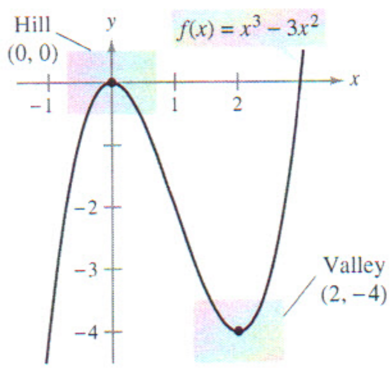
THEOREM 3.1 THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

DEFINITION OF RELATIVE EXTREMA

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum at $(c, f(c))$.**
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum at $(c, f(c))$.**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum,** respectively.

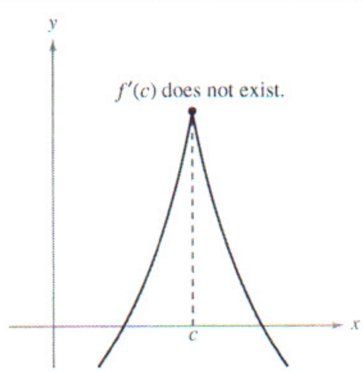


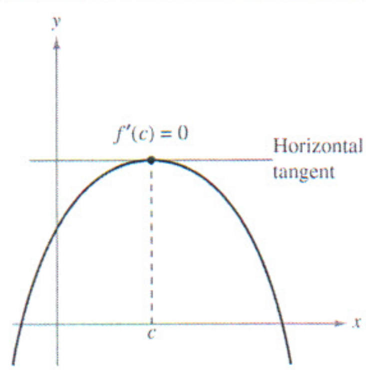
f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.

Informally, for a continuous function, you can think of a relative maximum as occurring on a "hill" on the graph, and a relative minimum as occurring in a "valley" on the graph.

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.





c is a critical number of f.

THEOREM 3.2 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

These "hills" and "valleys" can occur in two ways. If the function is smooth and rounded, the graph has a horizontal tangent line at the minimum or maximum. If it is sharp and peaked, the graph represents a function that is not differentiable ~~and~~ at the minimum or maximum.

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

1. Find the critical numbers of f in (a, b). - Set derivative = 0 and where DNE
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$.
   } find y-values for critical pts & endpts
4. The least of these values is the minimum. The greatest is the maximum.

Finding Extrema on a Closed Interval

$f(x) = 3x^4 - 4x^3$ *on the interval* $[-1,2]$

- find $f'(x) = 0$
- evaluate $f(x)$ at the critical numbers within the interval and the endpoints

$f(x) = 3x^4 - 4x^3$

$f' = 12x^3 - 12x^2 = 0$ Critical #: Set = 0

$12x^2(x-1) = 0$

$x = 0 \quad x = 1$

$[-1, 2]$

$f(-1) = 3 + 4 = 7$

$f(2) = 48 - 32 = 16$ Max.

$f(0) = 0$

$f(1) = 3 - 4 = -1$ Min.

Ans: $f(2) = 16$ Max

$f(1) = -1$ Min

$f(x) = 2x - 3x^{2/3}$ *on the interval of* $[-1,3]$

$f' = 2 - \cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{-1/3} = 2 - 2x^{-1/3} = 0$

$\cancel{2} = \cancel{2}x^{-1/3}$

$x^{1/3} = 1$

$x = 1$ AND $x = 0$

$f'(0) = DNE$

$[-1, 3]$

$f(-1) = -2 - 3 = -5$ Min

$f(1) = 2 - 3 = -1$

$f(3) = 6 - 3(3^{2/3})$

$= 6 - 3\sqrt[3]{9} \approx -.24$

$f(0) = 0$ Max

Ans:

Max: $f(0) = 0$, Min $f(1) = -5$

$f(x) = 2\sin x - \cos 2x$ *on the interval* $[0, 2\pi]$

$f' = 2\cos x + 2\sin 2x =$ (Double ∠ ID) $2\cos x + 2(2\sin x\cos x) = 0$

$2\cos x(1 + 2\sin x = 0$

$2\cos x = 0 \qquad 1 + 2\sin x = 0$

$\cos x = 0 \qquad \sin x = -\frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$f(0) = 2\sin 0 - \cos 2(0)$

$= 0 - 1 = -1$

$f(2\pi) = 0 - \cos 4\pi = -1$

$f\left(\frac{\pi}{2}\right) = 2(1) - \cos \cancel{2}\left(\frac{\pi}{\cancel{2}}\right)$

$= 2 - (-1) = 3$ MAX

$f\left(\frac{3\pi}{2}\right) = -2 - \cos 2\left(\frac{3\pi}{2}\right)$

$= -2 + 1 = -1$

$f\left(\frac{7\pi}{6}\right) = 2\left(-\frac{1}{2}\right) - \cos \cancel{2}\left(\frac{7\pi}{\cancel{6}_3}\right)$

$= -1 - \cos\frac{7\pi}{3}$ (Quad 1)

$= -1 - \frac{1}{2} = -\frac{3}{2}$ Min

$f\left(\frac{11\pi}{6}\right) = 2\left(-\frac{1}{2}\right) - \cos \cancel{2}\left(\frac{11\pi}{\cancel{6}_3}\right)$

$= -1 - \cos\frac{11\pi}{3}$ (Quad 4)

$= -1 - \frac{1}{2} = -\frac{3}{2}$ Min

Ans

Max: $f\left(\frac{\pi}{2}\right) = 3$

Min: $f\left(\frac{7\pi}{6}\right) = \frac{-3}{2}$

$f\left(\frac{11\pi}{6}\right) = \frac{-3}{2}$

3.2: Rolle's Theorem and the Mean Value Theorem

The Extreme Value Theorem from 3.1, states that on a closed interval, there will be both a minimum and a maximum, and they may occur on an endpoint. **Rolle's Theorem,** named after the French mathematician Michel Rolle (1652–1719), gives specific conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

THEOREM 3.3 ROLLE'S THEOREM

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

differentiable on end pts.

then there is at least one number c in (a, b) such that $f'(c) = 0$.

$f(x) = x^4 - 2x^2$ Find all values of c in the interval $(-2,2)$ such that $f'(c) = 0$

$f(-2) = 16 - 8 = 8$
$f(2) = 16 - 8 = 8$ $\} =$

$f' = 4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$x = 0 \quad x = \pm 1$

Derivative is 0 at each of these values. ∴ 3 extrema.

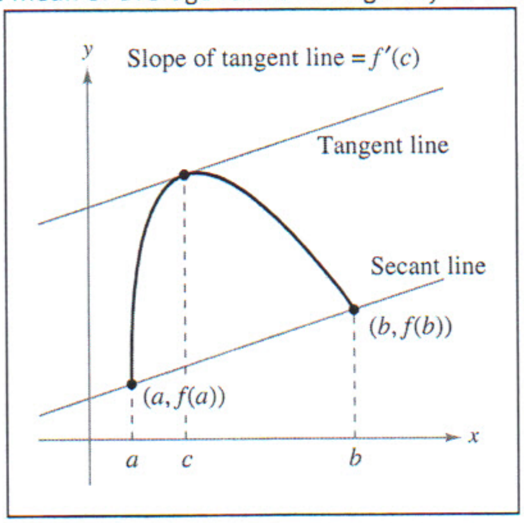
The Mean Value Theorem

THEOREM 3.4 THE MEAN VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The "mean" refers to the mean or average rate of change of f in the interval $[a, b]$



The Mean Value Theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$ For rate of change, ther must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the close interval $[a, b]$

Finding a Tangent Line

$f(x) = 5 - 4/x$ find all values of c in the open interval (1,4) such that

$f'(c) = \frac{f(4)-f(1)}{4-1} = \frac{4-1}{4-1} = 1 = \text{slope}$

in other words, to find the tangent line we will work backwards to find the value of f' and with the slope determine the value of x.

Given $f(x) = 5 - \frac{4}{x}$

slope $= f' = \frac{4}{x^2} = 1$

$4 = x^2 \quad \therefore x = \pm 2$

with interval (1,4), our point $\boxed{x = 2}$ or $c = 2$ is where we have a tangent line with slope $= 1$