Calculus
Lesson 2.4 The Chain Rule
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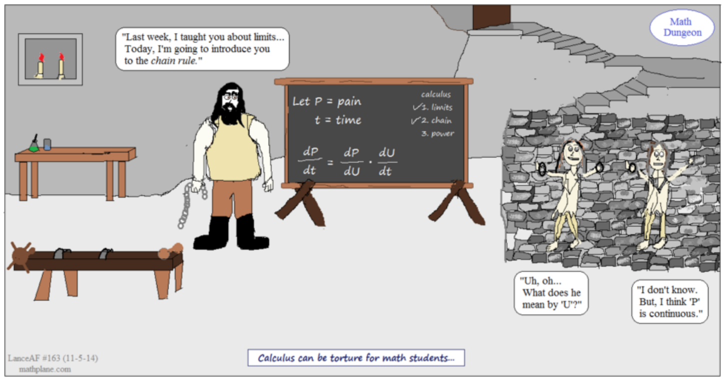

How do you take the derivative of: $F(x) = \sqrt{x^2+1}$???

Well, good question, maybe no answer? The differentiation formulas we have seen so far will not enable us to calculate $F'(x)$. F is a composite function. If we let the outside part be $y = f(u) = \sqrt{u}$ and the inside part be $u = g(x) = x^2 + 1$, we can then write:

$y = F(x) = f\big(g(x)\big) = f\ o\ g$. So!!! It would be nice to have a rule that tells us how to find the derivative of $F = f\ o\ g$. This is where the **chain rule** comes into play and it will find the derivative of F in terms of f *and* g! Fact is, the **chain rule** is one of the most important of the differentiation rules.

THEOREM 2.10 THE CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$		
b. $y = \sin 2x$		
c. $y = \sqrt{3x^2 - x + 1}$		
d. $y = \tan^2 x$		

Using the Chain Rule:

$y=\left(x^2+1\right)^3$

THEOREM 2.11 THE GENERAL POWER RULE

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n u^{n-1} u'.$$

Applying the General Power Rule:

Find the derivative:

$f(x)=\left(3x-2x^2\right)^3$

Differentiating Functions Involving Radicals

Find all points on the graph of f(x) for which f'(x)=0 and those for which f'(x) does not exist.

$f(x)=\sqrt[3]{\left(x^2-1\right)^2}$

Differentiating Quotients with Constant Numerators

$g(t)=\dfrac{-7}{(2t-3)^2}$

Simplifying by Factoring Out the Least Powers

$f(x) = x^2\sqrt{1-x^2}$

Simplifying the Derivative of a Quotient

$f(x) = \dfrac{x}{\sqrt[3]{x^2+4}}$

Simplifying the Derivative of a Power

$y = \left(\dfrac{3x-1}{x^2+3}\right)^2$

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

Applying the Chain Rule to Trigonometric Functions

a. $y = \sin 2x$ **b.** $y = \cos(x - 1)$ **c.** $y = \tan 3x$

Parentheses and Trigonometric Functions

a. $y = \cos 3x^2$ **b.** $y = (\cos 3)x^2$ **c.** $y = \cos(3x)^2$

d. $y = \cos^2 x$ **e.** $y = \sqrt{\cos x}$

Repeated Application of the Chain Rule

$f(t) = \sin^3 4t$

Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of f(x) at the point (π,1).

Then determine all values of x in the interval (0,2π) at which the graph of f has a horizontal tangent.

$f(x) = 2\sin x + \cos 2x$

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules

Let f, g, and u be differentiable functions of x.

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Sum or Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Product Rule:

$$\frac{d}{dx}[fg] = fg' + gf'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

Derivatives of Algebraic Functions

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\,u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}\,u'$$