Precalculus

Lesson 4.4: Properties of Rational Functions

Mrs. Snow, Instructor

When dealing with ratios of integers, they are identified as rational numbers. When we look at ratios of polynomials, we call them **rational functions.**

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$ A ratio of 2 polynomials

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Remember Denominator $\neq 0$

Find the domain of the rational functions:

$R(x) = \frac{2x^3 - 4}{x + 5} \neq 0$ $x+5 \neq 0$ $x \neq -5$ Interval notation Ans $(-\infty, -5) \cup (-5, \infty)$	$R(x) = \frac{1}{x^2 - 4} \neq 0$ $x^2-4 \neq 0$ $(x+2)(x-2) \neq 0$ $x+2 \neq 0$ $x-2 \neq 0$ $x \neq -2$ $x \neq 2$  Intervals Ans $(-\infty, -2) \cup (-2, 2) \cup (2-\infty)$

$$R(x) = \frac{x^3}{x^2 + 1 \neq 0}$$

$x^2+1 \neq 0$

$x^2 = -1$ what!?

$x \neq \sqrt{-1}$

x solutions/restrictions are that it cannot have imaginary values!

∴ Ans $(-\infty, \infty)$

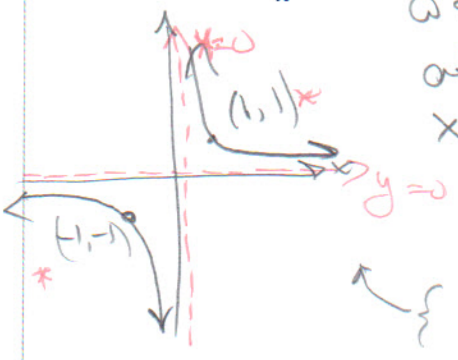
$$R(x) = \frac{x^2 - 1}{x - 1}$$

$x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

Graph and analyze. What happens at $x = 0$? As x gets closer to 0?
What happens as $x \to \infty$?

$$R(x) = \frac{1}{x}$$



Both have asymptotes at $x=0$ & $y=0$

$x \neq 0$ & $x \to \pm\infty$

$y \to 0$

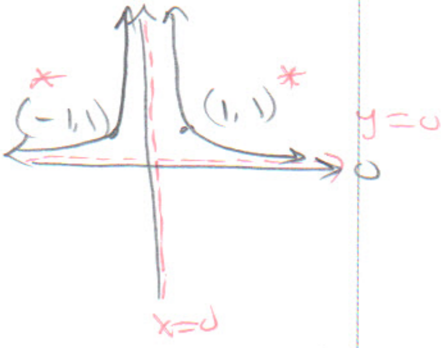
As $x \to 0$

$x = .001$

$y = \frac{1}{.001} = 1000$

critical points*

$$H(x) = \frac{1}{x^2}$$



$x \to 0$

$x = .001$

$y = 1000000$

Graph the rational function using transformations:

① parent

② transformations

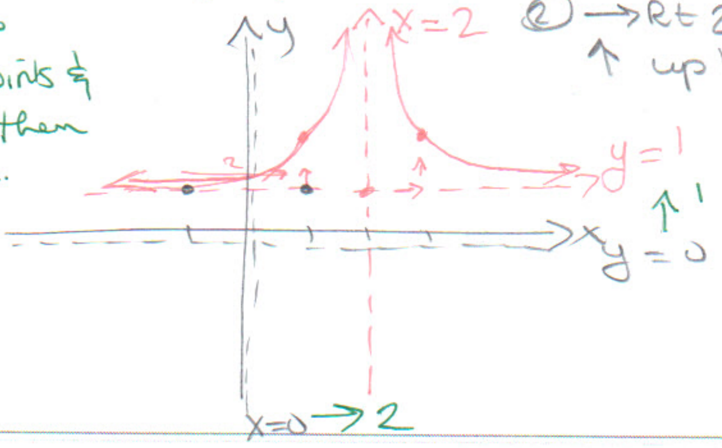
③ Move critical points & asymptotes then fill in curve.

$$R(x) = \frac{1}{(x-2)^2} + 1$$

① parent $\frac{1}{x2}$

② → Rt 2

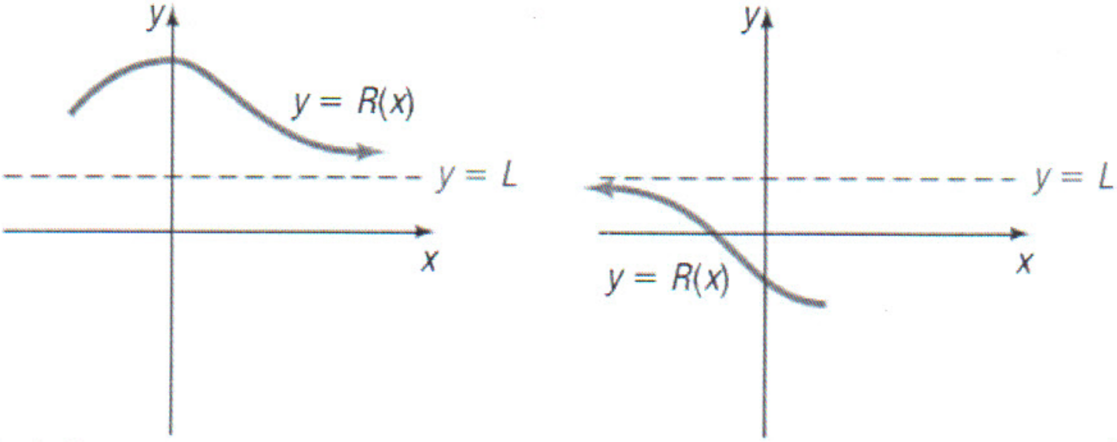
↑ up 1



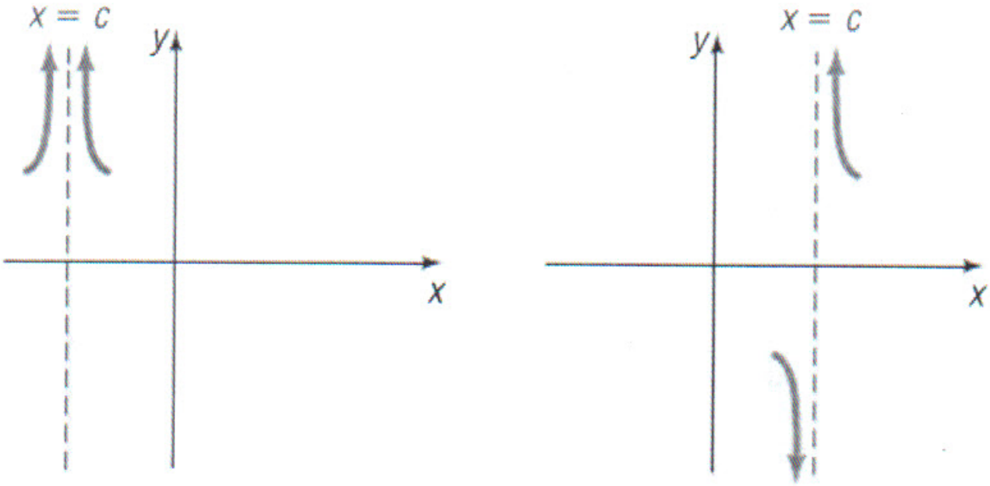
Asymptotes (see pg. 219 for more detail) *NOTE: Horizontal asymptotes may be intersected by the graph of a function! The graph of a function will never intersect a vertical asymptote.*

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.

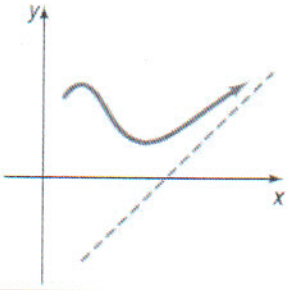
Horizontal asymptotes:



Vertical asymptotes:



There is also another type of asymptote, **OBLIQUE ASYMPTOTE**.



Vertical Asymptotes: For a rational function in lowest terms

- The values where the denominator goes to zero will be the vertical asymptotes; these are the domain restrictions and will graphically be seen as vertical asymptote(s).
- Factor denominator and set it equal to zero.

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \frac{5x^2}{3+x}$ $\quad 3+x=0$ $\quad x=-3$ $\quad$ VA: $x=-3$

(b) $R(x) = \frac{x}{x^2-4}$ $\quad (x+2)(x-2)=0$ $\quad x=-2,\ x=2$ $\quad$ V,A, $x=\pm 2$

(c) $H(x) = \frac{x^2}{x^2+1} = 0?$ $\quad$ No Restrictions $\quad$ NO VA

(d) $G(x) = \frac{x^2-9}{x^2+4x-21}$ $\quad \frac{(x+3)\cancel{(x-3)}}{(x+7)\cancel{(x-3)}}$ $\quad$ VA $x=-7$

$x=-7$

so what about $x=3$?
Hole!

Horizontal and Oblique Asymptotes

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

Horizontal Asymtotes

1. Degree of denominator is bigger: $m > n$ horizontal asymptote at $y = 0$
   BOBO
2. Degree of numerator is bigger: $n > m$ no horizontal asymptote *BUT....**
   BOTN
3. Degrees of numerator and denominator are equal $n = m$: Exponents are the same – divide leading coefficients to find the horizontal asymptote
   EATS DC

***Oblique Asymptote:**

When bigger on top, there will be an oblique asymptote. Divide the function. Quotient is the linear equation for the oblique asymptote.

$$r(x) = (ax+b) + \frac{r(x)}{q(x)}$$

There are 2 possibilities that we will explore:

1. Numerator Degree bigger by 1. **Asymptote is Quotient, the line $y = ax + b$**
2. Numerator Degree is bigger by more than 2.
   Quotient is a polynomial for degree 2 or higher.

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

degree-num: = 1
denom: = 2

Bigger on bottom

BOBO ← HA $\boxed{y=0}$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

deg = 4
deg = 3

Bigger on top → BOTN ← NOHA

OBlique!!

Quotient is Oblique Asymptote equation

$$\begin{array}{r} 3x+3 = y \\ x^3-x^2+1 \,\overline{)\,\cancel{3x^4} \qquad -x^2} \\ -\,\cancel{3x^4} - {}^{+}3x^3 + 3x \\ \hline 3x^3 - x^2 \\ 3x^2 - 3x^2 + 3 \\ \hline \underbrace{\qquad\qquad}_{\text{Remainder}} \end{array}$$

$$\boxed{y = 3x+3}$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$ EATS TC

Divide coefficients

HA $y = \frac{8}{4} = \boxed{2 = y}$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$ BOT N

no HA

Oblique Asymptote!

$2x^2 - 1 = y$

$x^3 - 1 \overline{) 2x^5 - x^3 + 2}$

$-\cancel{2x^5} \mp 2x^2$

$-\cancel{x^3} + 2x^2 + 2$

$\pm \cancel{x^3} \mp 1$

$\underbrace{\qquad}_{\text{Remainder}}$

Oblique Asymptote equation.

$\boxed{y = 2x^2 - 1}$

For more detail see textbook pg 224

SUMMARY

Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.
   (a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for very large values of $|x|$, the graph of R will behave like the graph of the quotient.

Note: The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote. ■

Lesson4.5: The Graph of a Rational Function

Calculators of course make graphing rational function much easier and quicker. However, we need to be proficient in using algebraic analysis to draw conclusions of the graph.

How to Analyze the Graph of a Rational Function

Step 1:
Factor the numerator and denominator or R. Find the domain of the rational function.
Step 2:
Write R in lowest terms.
Step 3:
Locate the intercepts of the graph.
Step 4:
Locate the vertical asymptotes. Graph each vertical asymptote using a dashed line.
Step 5:
Locate the horizontal or oblique asymptote, if one exists. Determine points in any at which the graph of R intersedts this asymotote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.
Step 6:
Graph R using a graphing calculator. Use the results in steps 1-5 to graph R by hand.

Analyze the rational function: ②

$$R(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x+2)(x-2)}$$

① D: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

③ Intercepts $(x,0)\ (0,y)$

$(x+2)\ (x-2)\ \frac{x-1}{\cancel{(x+2)(x-2)}} = 0\ \cancel{(x+2)(x-2)}$

$x - 1 = 0$

$\boxed{x = 1}$ $(1,0)$

$y = \frac{0-1}{0^2-4} = \frac{1}{4}$ $(0, \frac{1}{4})$

④ VA - Domain Restrictions

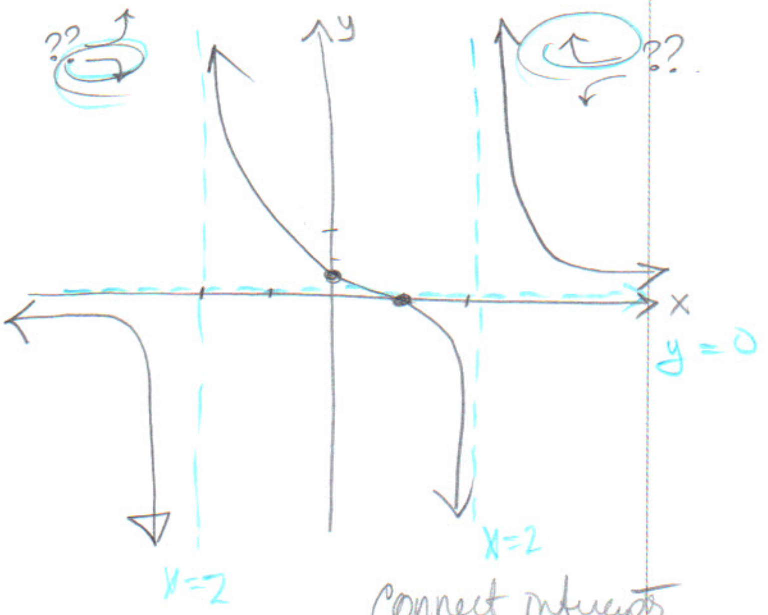
VA $x = -2,\ x = 2$

HA BOBO HA $y = 0$

Does $y \to +\infty$ or $-\infty$
as $x \to$ asymptotes

$X = -3$ $\frac{(-)}{(-)(-)} \therefore y \to -\infty$
Sign +/−

$X = 3$ $\frac{+}{(+)(+)}$
$y \to \infty$



Connect intercepts
follow trend

Analyze the rational function:

$$R(x) = \frac{x^2 - 1}{x}$$ ② Reduced

① D: $(-\infty,0)\cup(0,\infty)$

③ Intercepts $(x,0)$ $(\cancel{0,y})$

$y = \cancel{\frac{0^2-1}{0}}$ NO/undefined

$0 = \frac{(x^2-1))}{x}$

$0 = (x+1)(x-1)$ $(1,0)$

$x = 1, -1$ $(-1,0)$

④ VA at $x = 0$

HA BOTN → NONE

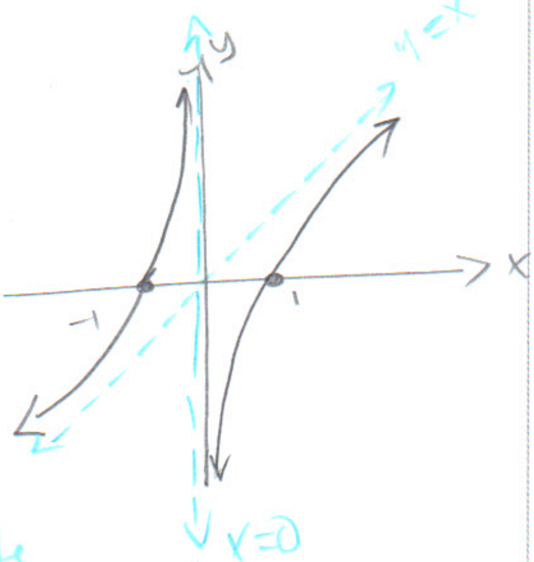
SO Divide

$x \overline{)\,x^2-1}$ quotient $x = y$

x^2

-1 Remainder

Oblique Asymptote



Analyze the rational function with a Hole

$$R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4} = \frac{(2x-1)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \frac{2x-1}{x+2}$$ ②

① D: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

③ Intercept $(x,0)$ $(0,y)$

$0 = 2x - 1$ $(\frac{1}{2},0)$, $(0,\frac{-1}{2})$

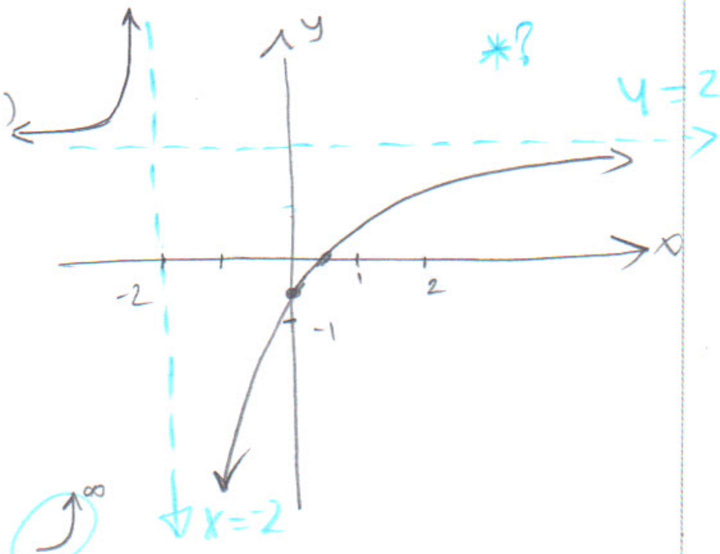
$\frac{1}{2} = x$

$y = \frac{2x-1}{x+2} = \frac{-1}{2}$

④ VA

$x = -2$

HA = EATSDC $y = 2$



∞

$x \to -2$

?

$-\infty$

let $x = -3$

$\frac{2x-1}{x+2} = \frac{-}{-} = +$

?* Will curve cross y=2? Ans: NO

$2 = \frac{2x-1}{x+2}$

$2(x+2) = 2x - 1$

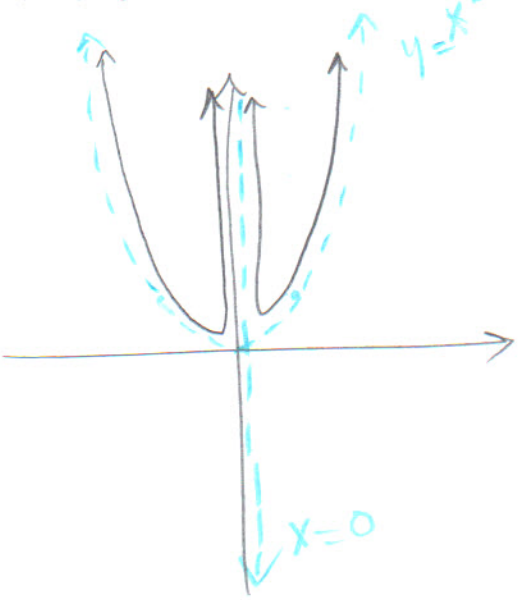
$\cancel{2x} + 4 = \cancel{2x} - 1$

$4 = -1$ False

Analyze the rational function:

$R(x) = \frac{x^4+1}{x^2}$ Reduced

① D: $(-\infty,\infty)$



② Interupts $(x,0)$ $(0,y)$
none None

$0 = x^4+1$

$-1 = x^4 \Rightarrow$ No sol

$y = \frac{0^4+1}{0}$ No sol

③ VA: $x=0$

HA: BOTN - NonE

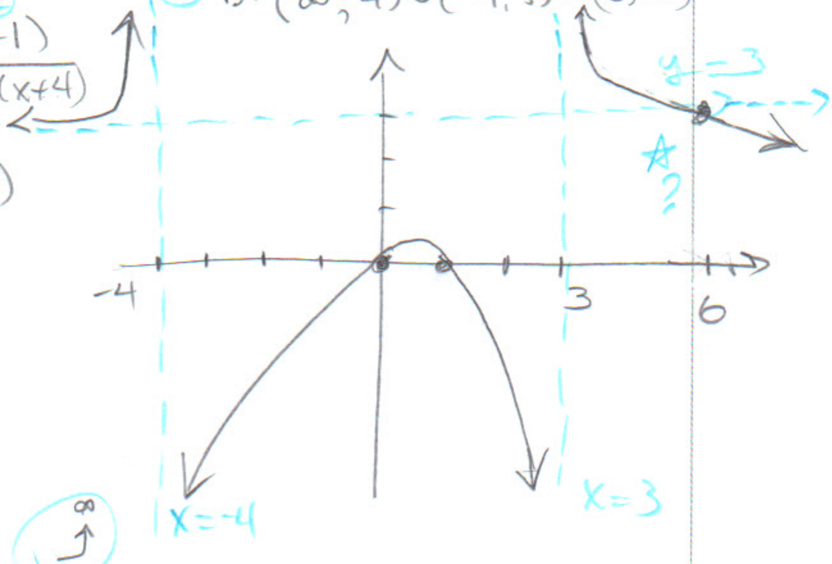
Oblique yes $y=x^2$

$$\begin{array}{r} x^2 = y \\ x^2 \overline{)\,\cancel{x^4}+1} \\ \underline{\cancel{x^4}} \\ 1 \text{ Remainder} \end{array}$$

Analyze the rational function:

$R(x) = \frac{3x^2-3x}{x^2+x-12}$ ② $= \frac{3x(x-1)}{(x-3)(x+4)}$

① D: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$



③ Intercepts $(x,0)(0,y)$

$0 = 3x(x-1)$ $(0,0)$

$x=0$ $x=1$ $(1,0)$

$y = \frac{3(0)(0-1)}{(0-3)(0+4)} = 0$

④ VA $x=3$ $x=-4$

HA EATSDC $y=3$

∞ ↑ $x \to -4$? ↓ $-\infty$

$x=-5$

$\frac{(-)(-)}{(-)(-)} = +$

∞ ↑ $x \to 3$? ↓ $-\infty$

$x=4$

$\frac{(+)(+)}{(+)(+)}$ +

☆ ? Will curve cross $y=3$ Asympt? Yes

$3 = \frac{3x^2-3x}{x^2+x-12}$

$\cancel{3x^2}+3x-36 = \cancel{3x^2}-3x$

$6x = 36$, $\boxed{x=6}$

crosses at $x=6$

Finding the Least Cost

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters (cm^3), or $\frac{1}{2}$ liter. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢/per square centimeter (cm^2). The sides of the can are made of material that costs 0.02¢/cm^2.

VOLUME

(a) Express the cost of material for the can as a function of the radius r of the can.

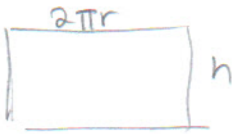
(b) Use a graphing utility to graph the function $C = C(r)$.

(c) What value of r will result in the least cost?

(d) What is this least cost?

$A_{\text{Top + Bottom}} = (\pi r^2)2$

$A_{\text{Side}} = 2\pi r h$



Cost = cents

$C = .05(2\pi r^2) + .02(2\pi r h)^{*}$

$= .05(2\pi r^2) + .02(2\cancel{\pi}\cancel{r}(\frac{500}{\cancel{\pi} r^2}))$

$= .1\pi r^2 + \frac{20}{r}$

common denom

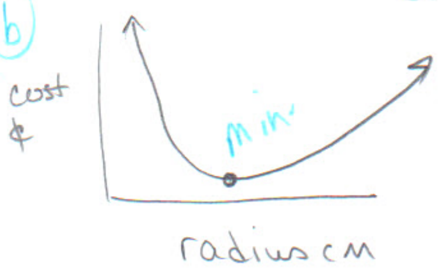
ⓐ $= \frac{.1\pi r^3}{r} + \frac{20}{r} = \frac{.1\pi r^3 + 20}{r} = \text{Cost}$

$\text{Volume} = 500 = \pi r^2 h$

substitute $\frac{500}{\pi r^2} = h^{*}$

Graph

ⓑ



ⓒ ⓓ Least cost is ¢ = 9.47 cents

Least cost happens when radius = 3.17 cm