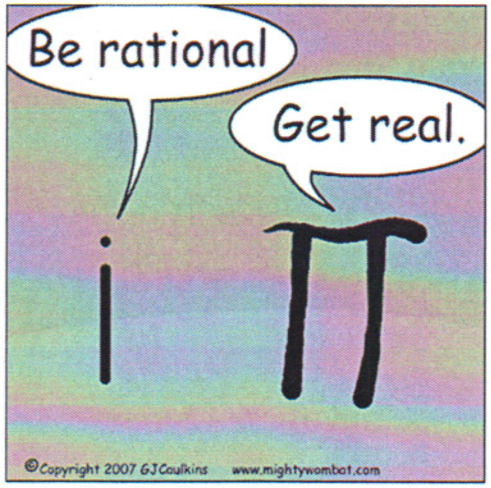
Precalculus

Lesson 4.2: The Real Zeros of a Polynomial Function

Mrs. Snow, Instructor



Dividing polynomials is a very similar process to the old long divisions we did in elementary school:

$$842 \div 15 = ???$$

56 r 2 or $56\frac{2}{15}$

$$\begin{array}{r} 56 \\ 15\overline{)842} \\ -75 \\ \hline 92 \\ 90 \\ \hline 2 \end{array}$$

Proper format for division of polynomials:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad f(x) = q(x)g(x) + r(x)$$

$f(x)$ ↑ dividend, $q(x)$ ↑ quotient, $g(x)$ ↑ divisor, $r(x)$ ↑ remainder

Divide:

$6x^2 - 26x + 12 \; by \; x - 4$

$$\begin{array}{r} 6x - 2 \\ x-4 \,\big)\, 6x^2 - 26x + 12 \\ -(6x^2 - 24x) \\ \hline -2x + 12 \\ -(-2x + 8) \\ \hline 4 \end{array}$$

$f(x) = (6x-2)(x-4) + 4$

check:

$6x^2 - 26x + 8 + 4 =$

$6x^2 - 26x + 12$ ✓

Divide using synthetic division; note this works only for divisors in the form of $\boldsymbol{x - c}$:

$2x^3 - 7x^2 + 5 \quad by \quad x - 3$

$x^3 \quad x^2 \quad x^1 \quad c \qquad c = 3$

3	2	-7	0	5
	↓	6	-3	-9
	2	-1	-3	-4

coefficients polynomial 1 degree less; remainder

$f(x) = (2x^2 - x - 3)(x-3) - 4$

The Remainder Theorem

If the polynomial f(x) is divided by $x - c$, then the remainder, $\boldsymbol{r(x)}$, is the value $f(c)$.

while we could use division to see if there is a remainder, our theorem says that the ***remainder*** $\boldsymbol{r(x) = f(c)}$

Using the Remainder Theorem, to find the remainder of:

$f(x) = x^3 - 4x^2 - 5$ (you can check your work with division).

1. $x - 3$, $c = 3$ $\qquad$ $x + 2$, $c = -2$

$f(3) = 3^3 - 4(3^2) - 5$

$= 27 - 36 - 5$

$= -14$ = Remainder

$f(-2) = (-2)^3 - 4(-2)^2 - 5$

$= -8 - 16 - 5$

$= -29$ = Remainder

Let's take the remainder theorem a step farther. If $r(x) = 0$, what does that tell us about the factor, $(x - c)$??

(x-c) would be a factor of our function

Factor Theorem

Let f be a polynomial function. $(x - c)$ *is a factor of* $f(x)$ *if and only if* $f(c) = 0$

When $f(c) = 0$*, the remainder is 0, therefore,* $(x - c)$ *is a factor*

Use the factor theorem to determine whether the function

$$f(x) = 2x^3 - x^2 + 2x - 3$$

has the factor

a) $x - 1$, $c = 1$

$f(1) = 2(1^3) - (1^2) + 2(1) - 3$
$= 2 - 1 + 2 - 3 = 0$

yes, $x-1$ is a factor

b) $x + 2$, $c = -2$

$f(2) = -16 - 4 - 4 - 3$
$= -27 \neq 0$

$(x+2)$ not a factor

Finding the roots

Take a look at $P(x) = (x - 2)(x - 3)(x + 4)$ multiplying the factors together we get:

$P(x) = x^3 - x^2 - 14x + 24$ Where did the constant 24 come from?

So, the constants of the factors multiplied out give us the constant of P(x). If the product of the zeros equals the constant then the zeros are all factors of the constant! We use this fact in the form of the **Rational Zeros Theorem**.

Rational Zeros Theorem

If the polynomial, P, has integer coefficients,

then every rational zero of P is of the form $\pm\frac{p}{q}$

q= is a factor of the leading coefficient

$$f(x) = \mathbf{3}x^3 + 8x^2 - 7x - \mathbf{12}$$

p= is a factor of the constant

After finding all the possible rational roots, one simply uses the factor theorem to determine if the number is in fact a root. Graphing calculators make this process much easier. Remember! If you are not allowed to use a graphing calculator, you will want to plug all values into the polynomial to determine which are roots.

List the potential rational zeros of:

$q \quad p$

$$f(x) = 3x^3 + 8x^2 - 7x - 12$$

$\pm p = \pm\{1, 2, 3, 4, 6, 12\}$

$q = \pm\{1, 3\}$

$\frac{p}{q} = \pm\left\{1, 2, 3, 4, 6, 12, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right\}$

List all p's over 1st q, then all p's over 2nd q and so on. Reduce so not to have duplicates.

From your textbook, pg 203

SUMMARY Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 2 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

NUMBER OF REAL ZEROS

A polynomial function cannot have more real zeros than its degree.

Solve by factoring:

$x^2 + 13x + 36 = 0$

$(x+9)(x+4) = 0$

$x+9=0 \quad x+4=0$

$x = -9 \quad x = -4$

Products = 36 & add up to 13

$2x^2 - 5x - 7 = 0$

$2x^2 - 7x + 2x - 7 \quad (2)(-7) = -14$

$x(2x-7) + 1(2x+7 \quad 2 - 7 = -5$

$(x+1)(2x+7) = 0$

$x+1=0 \quad 2x+7=0$

$x = -1 \quad x = -\frac{7}{2}$

1. List the possible rational zeros for the function below.
2. Using synthetic division, find the real zeros of the polynomial.
3. Factor completely.

$f(x) = 3x^3 + 8x^2 - 7x - 12$ you will find that: $(x = -1)$

① $p \pm \{1, 2, 3, 4, 6, 12\}$

$q \pm \{1, 3\}$

$\frac{p}{q} = \pm \{1, 2, 3, 4, 6, 12, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\}$

with $x = -1$ synthetic division

$$\begin{array}{r|rrrr} -1 & 3 & 8 & -7 & -12 \\ & & -3 & -5 & 12 \\ \hline & 3 & 5 & -12 & \boxed{0} \end{array}$$

$x = -1$ is a zero so should not have a remainder

$(x+1)(3x^2 + 5x - 12)$ ← factor

$3x^2 - 4x + 9x - 12$ factor by grouping

-36

$(4)(9)$

$-4 + 9$

$x(3x-4) + 3(3x-4)$

$(x+1)(x+3)(3x-4) = f(x)$

Zeros: $x = -1$, $x = -3$, $x = \frac{4}{3}$

For the following function:
List all possible rational roots
Find the real zeros of f(x) and write in factored form:

$f(x) = 2x^4 + 13x^3 + 29x^2 + 27x + 9$ $\qquad x = -1, -3$

$p = \pm\{1, 3, 9\}$ $\qquad \frac{p}{q} = \pm\{1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}\}$

$q = \pm\{1, 2\}$

$x = -1, -3$ Synthetic division

-1	2	13	29	27	9
		-2	-11	-18	-9
-3	2	11	18	9	0
		-6	-15	-9	
	2	5	3	0	

$\Rightarrow 2x^3 + 11x^2 + 18x + 9$

$\rightarrow 2x^2 + 5x + 3 = (2x+3)(x+1)$

Factored Form

$f(x) = (x+1)(x+3)(2x+3)(x+1)$ Oh!

$= (x+1)^2(x+3)(2x+3)$ ← simplify

(multiplicity = 2)
touch x-axis ☺

Zeros: $x = -1, -3, -\frac{3}{2}$

Factor by grouping to find the zeros of the function (no calculator!!!):

$$f(x) = 4x^3 - 32x^2 - x + 8$$

$4x^2(x-8) - 1(x-8)$

$(4x^2-1)(x-8) = 0$

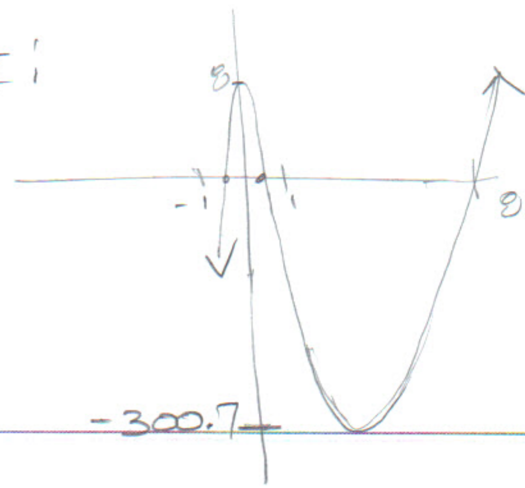
$4x^2 - 1 = 0$ $\quad$ $x - 8 = 0$

$4x^2 = 1$ $\quad$ $x = 8$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

FYI:



Precalculus

Lesson: 4.3 Complex Zeros; Fundamental Theorem of Algebra

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Not all quadratic equations have real solutions.

A variable in the complex number system is referred to as a **complex variable.** A **complex polynomial function** f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **complex zero** of f if $f(r) = 0$.

If we look at the complex number system, every quadratic equation has at least one complex solution; remember rational and irrational roots are in fact complex numbers. We just don't write them in the complex form. The fact that each polynomial function will have a complex solution brings about an important theorem.

Fundamental Theorem of Algebra.

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

Another important theorem states that if we have the solution $a + bi$ then we must also have the solution $a - bi$.

Conjugate Pairs Theorem.

Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

and a corollary

A polynomial function f of odd degree with real coefficients has at least one real zero.

Using the Conjugate Pairs Theorem

A polynomial function f of degree 5 whose coefficients are real numbers has the zeros 1, $5i$, and $1 + i$. Find the remaining two zeros.

degree 5 ⇒ 5 zeros

$x = 1$ $\quad$ $x = 5i$ $\quad$ $x = 1 + i$

conjugate $\quad$ conjugate

$x = -5i$ $\quad$ $x = 1 - i$

Example:

Find a polynomial function of degree 4 whose coefficients are real numbers and that has the zeros of $1, 1, -4 + i$

zeros $x = 1$ $x = 1$ $x = -4 + i$ AND $x = -4 - i$

factors $x - 1 = 0$ $x + 4 - i = 0$ $x + 4 + i = 0$

$x - 1 = 0$

$f(x) = (x-1)(x-1)(x+4-i)(x+4+i)$

$= (x^2 - 2x + 1)(x^2 + 4x + xi + 4x + 16 + 4i - xi - 4i - i^2)$

$+1$

$= (x^2 - 2x + 1)(x^2 + 8x + 16 + 1)$

$f(x) = (x^2 - 2x + 1)(x^2 + 8x + 17)$

$x^4 + 8x^3 + 17x^2$

$-2x^3 - 16x^2 - 34x$

$x^2 + 8x + 17$

$f(x) = x^4 + 6x^3 + 2x^2 - 26x + 17$ Ans

$-i^2 = ?$

by definition

$i^2 = -1$

so

$-(i^2) = -(-1)$

$-i^2 = 1$

#1 ID all zeros

#2 turn zeros into factors

#3 multiply real factors & multiply imaginary factors separately

#4 all "i" values should simplify out

#5 finish multiplying.

Given that $x = \pm 2$ are solutions, find the remaining complex zeros of the polynomial function.

$$f(x) = x^4 + 2x^3 + x^2 - 8x - 20$$

Synthetic division

-2	1	2	1	-8	-20	factor
		-2	0	-2	20	
2	1	0	1	-10	0	⇐ $(x^3 + x - 10)$
		2	4	10		
	1	2	5	0		⇐ $(x^2 + 2x + 5)$

So:

$f(x) = (x-2)(x+2)(x^2+2x+5)$

↑ cannot factor
use Quadratic formula

$$x = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$= \frac{2(-1 \pm 2i)}{2} = x$$

Always Reduce!

Zeros: $x = \pm 2$
$x = -1 \pm 2i$ } Ans

Don't forget the 2 zeros you were given