Calculus

Lesson 2.3: Product and Quotient Rules and Higher-Order Derivatives

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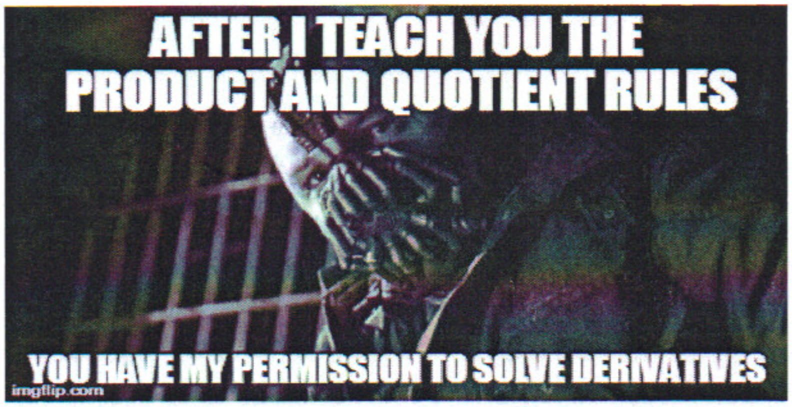


While we now know that the derivative of the sum of two functions is simply the sum of their derivatives, the rules change for the product and quotient of two functions.

**THEOREM 2.7 THE PRODUCT RULE**

The product of two differentiable functions $f$ and $g$ is itself differentiable. Moreover, the derivative of $fg$ is the first function times the derivative of the second, plus the second function times the derivative of the first. How I learned it:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$ Derivative of 1st x 2nd plus Derivative of 2nd x 1st

**Using the Product Rule**

Find the derivative of:

$h(x) = (3x - 2x^2)(5 + 4x)$.

$h' = 4(3x - 2x^2) + (3 - 4x)(5 + 4x)$

$= 12x - 8x^2 + 15 - 8x - 16x^2$

$\boxed{= -24x^2 + 4x + 15}$

Find the derivative of:

$y = 3x^2 \sin x$

$y' = (\cos x)(3x^2) + 6x \operatorname{sen} x$ (factoring)

$= \boxed{3x(x\cos x + 2\sin x)}$

Find the derivative of:
$y = 2x\cos x - 2\sin x$

$y' = (-\sin x)(2x) + 2\cancel{\cos x} - \cancel{2\cos x}$

$y' = \boxed{-2x\sin x}$

**THEOREM 2.8 THE QUOTIENT RULE**

The quotient $f/g$ of two differentiable functions $f$ and $g$ is itself differentiable at all values of $x$ for which $g(x) \neq 0$. Moreover, the derivative of $f/g$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator. derivative of 1st x 2nd minus derivative 2nd x 1st / (2nd)²

1st / 2nd

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

As previous sections we rewrite
Do so here too

**Using the Quotient Rule**

Find the derivative of:
$y = \dfrac{5x-2}{x^2+1}$

$$y' = \frac{(5)(x^2+1) - 2x(5x-2)}{(x^2+1)^2}$$

$$= \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$$

$$= \boxed{\frac{-5x^2+4x+5}{(x^2+1)^2}}$$

Find the derivative by first rewriting:

$f(x) = \dfrac{x\left(3 - \left(\frac{1}{x}\right)\right)}{x(x+5)}$ multiply by $\frac{x}{x}$ so to remove $\frac{1}{x}$

$$f(x) = \frac{3x-1}{x^2+5x}$$

$$f' = \frac{3(x^2+5x) - (2x+5)(3x-1)}{(x^2+5x)^2}$$

$$= \frac{3x^2+15x-(6x^2+12x-5)}{(x^2+5x)^2}$$

$$= \frac{3x^2+15x-6x^2-12x+5}{(x^2+5x)^2}$$

$$= \boxed{\frac{-3x^2+3x+5}{(x^2+5x)^2}}$$

**Using the Constant Multiple Rule**

Original Function: $y = \frac{x^2+3x}{6}$

Rewrite: $y' = \frac{1}{6}(x^2+3)$

Now Differentiate: $y' = \frac{1}{6}(2x+3)$

And Simplify: $\frac{2x+3}{6}$

Factor!

$y = \frac{5x^4}{8} = \frac{5}{8}x^4$

$y' = 4\left(\frac{5}{8}\right)x^3$

$y' = \boxed{\frac{5}{2}x^3}$

$y = \frac{-3(3x-2x^2)}{7x} = \frac{-3\cancel{x}(3-2x)}{7\cancel{x}}$

$y = \frac{-3}{7}(3-2x)$

$y' = -\frac{3}{7}(-2) = \boxed{\frac{6}{7}}$

$y = \frac{9}{5x^2} = \frac{9}{5}x^{-2}$

$y' = -2\left(\frac{9}{5}\right)(x^{-3}$

$= \frac{-18}{5x^3}$

**THEOREM 2.9 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS**

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Note: Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text. So, we need to be careful when comparing answers. One simply cannot assume that an answer is correct, you will want to verify that is variation of the correct answer.

!!

Two characteristics of a simplified form are the absence of negative exponents and the combining of like terms.

**Applying the Quotient Rule**

Show that the derivative is true: Prove:

$\frac{d}{dx}[\tan x] = sec^2$

LHS

$\frac{d}{dx} \frac{\sin x}{\cos x} =$

$\frac{(\cos x)(\cos x) + \sin x(\sin x)}{\cos^2 x} =$

$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} =$

$\frac{1}{\cos^2 x} =$

$\sec^2 x = \text{RHS}$

QED

Differentiate the following Trigonometric Functions

$y = x - \tan x$

$y' = \boxed{1 - \sec^2 x}$

$y = x\sec x$

$y' = (\sec x \tan x)(x) + \sec x$

$y' = \boxed{\sec x(1 + x\tan x)}$

**Different Forms of a Derivative**

Differentiate both forms:

Since both sides are equal, the derivatives of both sides will be equal too.

$$y = \frac{1 - \cos x}{\sin x} = \csc x - \cot x$$

$\frac{d}{dx}\left(\frac{1 - \cos x}{\sin x}\right) =$

$= \frac{\sin x \sin x - \cos x(1 - \cos x)}{\sin^2 x}$

$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$

$= \frac{1 - \cos x}{\sin^2 x}$ <=>

$\frac{d}{dx}(\csc x - \cot x) =$

$-\csc x \cot x + \csc^2 x$

$\frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$

$\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$

$\frac{1 - \cos x}{\sin^2 x}$

LHS = RHS

**Higher-Order Derivatives**

| | |
|---|---|
| $s(t)$ | Position function |
| $v(t) = s'(t)$ | Velocity function |
| $a(t) = v'(t) = s''(t)$ | Acceleration function |

**Finding the Acceleration Due to Gravity**

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by:

$$s(t) = -0.81t^2 + 2$$

where *s*(*t*) is the height in meters and *t* is the time in seconds. What is the acceleration due to gravity on the moon?

To find the acceleration, differentiate the position function twice:

velocity $s'(t) = -1.62t$

acceleration $s''(t) = -1.62$ meters/sec.