Calculus

Lesson 2.2: Basic Differentiation Rules and Rate of Change

Mrs. Snow, Instructor

DON'T DRINK AND DERIVE

Mathematicians
Against
Drunk
Deriving

Know Your Limits!

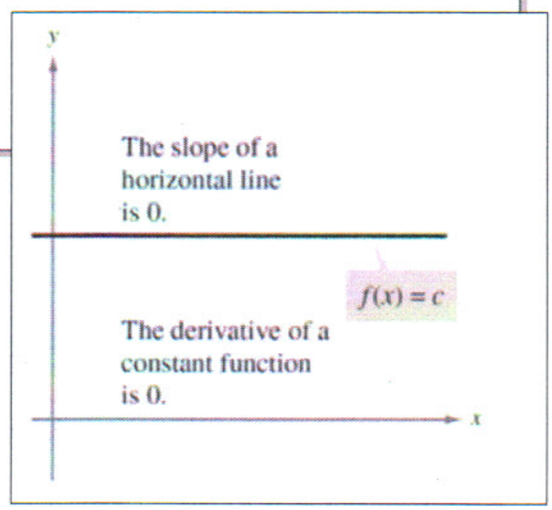
In 2.1 we reviewed the limit definition to find derivatives. Now we are ready move on to several "differentiation rules" that allow you to find the derivatives without the *direct* use of the limit definition!

THEOREM 2.2 THE CONSTANT RULE

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

(See Figure 2.14.)



Using the Constant Rule:

Function	Derivative
$y = 7$	$\frac{dy}{dx} = 0$
$f(x) = 0$	$f'(x) = 0$
$s(t) = -3$	$s'(t) = 0$
$y = k\pi^2$ (k is constant)	$y' = 0$

THEOREM 2.3 THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Using the power rule

Function	*Derivative*
a. $f(x) = x^3$	$f' = 3x^{3-1} = \boxed{3x^2}$
b. $g(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$	$g' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \boxed{\frac{1}{3x^{2/3}}}$
c. $y = \frac{1}{x^2} = x^{-2}$	$y' = -2x^{-3} = \boxed{-2/x^3}$

Finding the Slope of the Graph:

Find the slope of the graph of $f(x) = x^4$ when: $f' = 4x^3$

a. $x = -1$

$m = 4(-1)^3$
$= -4$

b. $x = 0$

$m = 4(0^3)$
$m = 0$

c. $x = 1$

$m = 4(1^3)$
$m = 4$

Finding an Equation of a Tangent Line

Find and equation of the tangent line to the graph of $f(x) = x^2$ when $x = 2$, $y = 4$

$f' = 2x$

$m = 2(2) = 4$

$y = mx + b$

$4 = 4(2) + b$

$4 - 8 = \boxed{b = -4}$

$y = 4x - 4$

OR point slope form:

$y_1 - y = m(x_1 - x)$

$y - 4 = 4(x - 2)$ $+4$ $+4$

$y = 4x - 8$

$\boxed{y = 4x - 4}$

THEOREM 2.4 THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

Using the Constant Multiple Rule

$y = \frac{2}{x} = 2(x^{-1})$

$\frac{dy}{dx} = -2x^{-2}$

$= \frac{-2}{x^2}$

$f(t) = \frac{4t^2}{5}$ $\rightarrow f' = \frac{4}{5}(2)(x)$

$= \frac{8}{5}x$

$y = 2\sqrt{x} = 2x^{1/2}$

$y' = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

$y = \frac{1}{2\sqrt[3]{x^2}} = \frac{1}{2}x^{-\frac{2}{3}}$

$y' = \frac{1}{2}\left(\frac{-2}{3}\right)x^{-5/3}$

$y' = \frac{-1}{3x^{5/3}}$

$y = -\frac{3x}{2}$

$y' = -\frac{3}{2}x^{1-1}$

$= -\frac{3}{2}$

Using Parentheses When Differentiating

Be careful when solving a problem:

$y = \frac{7}{3x^{-2}} = \frac{7}{3}x^2$

$y' = \frac{14}{3}x$

vs.

$y = \frac{7}{(3x)^{-2}} = 7(3x)^2 = 7(9x^2) =$

$= 63x^2$

$y' = 126x$

$y = \frac{5}{2x^3} = \frac{5}{2}x^{-3}$

$y' = (-3)\left(\frac{5}{2}\right)x^{-4}$

$= \frac{-15}{2x^4}$

vs.

$y = \frac{5}{(2x)^3} = \frac{5}{8x^3} = \frac{5}{8}x^{-3}$

$y' = \frac{-15}{8x^4}$

THEOREM 2.5 THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Using the Sum and Difference Rules

Function	*Derivative*
a. $f(x) = x^3 - 4x + 5$	$f' = 3x^2 - 4$
b. $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g' = \frac{-4x^3}{2} + 9x - 2$ $= \boxed{-2x^3 + 9x - 2}$

Derivatives of Sine and Cosine Functions

We also have formulas for our trig functions, two of which are below:

THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

$y = 2\sin x$

$y' = 2\cos x$

$y = \frac{\sin x}{2} = \frac{1}{2}\sin x$

$y' = \frac{1}{2}\cos x$

OR $= \frac{\cos x}{2}$

$y = x + \cos x$

$y' = 1 - \sin x$

Rates of Change

$$Average\ velocity = \frac{Change\ in\ Distance}{Change\ in\ Time} = \frac{\Delta y}{\Delta x}$$

Make sure to distinguish between instantaneous and average velocity. All else fails read the directions!

If a Billiard ball is dropped from a height of 100 ft, its height s at time t is given by $s = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

$a)$ $[1, 2]$

$S(1) = -16(1^2)+100$
$= 84'$
$S(2) = -16(4)+100$
$= 36'$

$$\frac{\Delta s}{\Delta t} = \frac{36-84}{2-1}$$
$$\boxed{= -48\frac{ft}{sec}}$$

$b)$ $[1, 1.5]$

$S(1) = 84'$
$S(1.5) = -16(1.5^2)+100$
$= 64'$

$$\frac{\Delta s}{\Delta t} = \frac{64-84}{1.5-1}$$
$$\boxed{= -40\frac{ft}{sec}}$$

$c)$ $[1, 1.1]$

$S(1.1) =$
$-16(1.1^2)+100$
$= 80.64'$

$$\frac{\Delta s}{\Delta t} = \frac{80.64-84}{1.1-1}$$
$$\boxed{= -33.6\frac{ft}{sec}}$$

THEOREM 2.5 THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Using the Sum and Difference Rules

Function	*Derivative*
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

Rates of Change

$$\textit{Average velocity} = \frac{\textit{Change in Distance}}{\textit{Change in Time}} = \frac{\Delta y}{\Delta x}$$

Make sure to distinguish between instantaneous and average velocity. All else fails read the directions!

If a Billiard ball is dropped from a height of 100 ft, its height s at time t is given by $s = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

$a)$ $[1, 2]$

$S(1) = -16 + 100 = 84'$

$S(2) = -16(2^2) + 100 = 36'$

$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1}$

$\boxed{= -48 \frac{ft}{sec.}}$

$b)$ $[1, 1.5]$

$S(1) = 84'$

$S(1.5) = 64$

$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1}$

$\boxed{= -40 \frac{ft}{sec}}$

$c)$ $[1, 1.1]$

$S(1) = 84'$

$S(1.1) = 80.64'$

$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1}$

$= \boxed{-33.6 \frac{ft}{sec}}$

Remember, slope is rate of change. If the rate of change is distance per time, then the slope is a velocity. Since the derivative is the rate of change at a point P, we have an instantaneous rate of change or an instantaneous velocity:

Instantaneous Velocity

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = s'(t)$$

where the position function is given by:

$$s(t) = \frac{1}{2}gt^2 + v_o t + s_o$$

So is the initial height of the object, v_o is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately $-32\ feet\ per\ second$ or $-9.8\ meters\ per\ second.$

Using the derivative to Find Velocity

- At time t=0, a diver jumps from a platform diving board that is 32 feet above the water.
- The position of the diver is given by $\boldsymbol{s(t) = -16t^2 + 16t + 32}$ where s is measured in feet and t is measured in seconds.
  a. When does the diver hit the water? — height = 0 so....
  b. What is the diver's velocity at impact?

a

$0 = -16t^2 + 16t + 32$

$0 = -16(t^2 - t - 2)$

$0 = -16(t-2)(t+1)$

$\boxed{t = 2 \text{ sec}}$ $\cancel{t = -1}$

Velocity $= S' = -32t + 16$

at $t = 2$ $S' = -32(2) + 16$

$64 + 16 = \boxed{-48 \frac{ft}{sec}}$