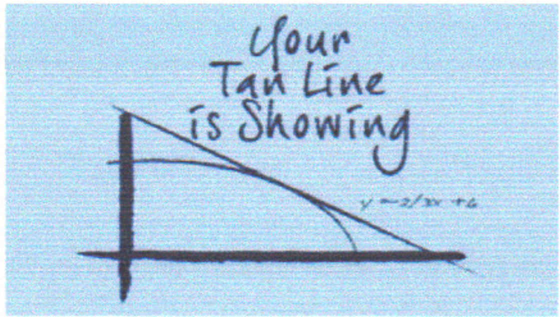
Calculus
Lesson 2.1 The Derivative and the Tangent Line Problem
Mrs. Snow, Instructor



The 411 of calculus pretty much goes back to the 17th century to several problems mathematicians were facing. In this section we are looking at the tangent line problem. To find a tangent line at a point P, what we end up having to do is to find the slope of the tangent line at point P. We start out with the slope of a secant line and as the change in the horizontal distance Δx, approaches 0, we can obtain a more and more accurate slope approximation for the secant line which in turn becomes a line tangent at a point P.

Definition of Tangent Line with Slope *m*

If f is defined on an open interval containing c, and if the limit (***a.k.a.***)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m \quad = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Find the slope of the graph of $f(x)$ at the point $(2,1)$. point $(a, f(a))$

$f(x) = 2x - 3$

$a + h = (2+h)$

$$\lim_{h \to 0} \frac{2(2+h) - 3 - 1}{h} =$$

$$\lim_{h \to 0} \frac{\cancel{4} + 2\cancel{h} - \cancel{3} - \cancel{1}}{\cancel{h}} \quad \boxed{= 2 = m}$$

Find the slopes of the tangent lines to the graph of $f(x)$ at the points $(0,1)$ *and* $(-1,2)$

$f(x) = x^2 + 1$

find limit of $(a, f(a))$

$$\lim_{h \to 0} \frac{(a+h)^2 + 1 - (a^2 + 1)}{h} =$$

$$\lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 + \cancel{1} - \cancel{a^2} - \cancel{1}}{h} =$$

$$\lim_{h \to 0} \frac{\cancel{h}(2a + \cancel{h} \to 0)}{\cancel{h}} = \boxed{2a = m}$$

at $(0,1)$

$\boxed{m = 2(0) = 0}$

at $(-1,2)$

$\boxed{m = 2(-1) = -2}$

Definition of the Derivative of a Function

The **derivative** of f at x is given by

$f'(x) =$ $$f'(x) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Some common notations for the derivative:

$$f'(x),\quad \frac{dy}{dx},\quad y',\quad \frac{d}{dx}[f(x)],\quad D_x[y].$$

Find the derivative of $f(x)$ where:

$f(x) = x^3 + 2x$

Binomial coefficients
1-3-3-1

$$\lim_{h\to 0}\frac{(a+h)^3+2(a+h)-(a^3+2a)}{h} =$$

$$\lim_{h\to 0}\frac{\cancel{a^3}+3a^2h+3ah^2+h^3+\cancel{2a}+2h-\cancel{a^3}-\cancel{2a}}{h}$$

$$\lim_{h\to 0}\frac{\cancel{h}(3a^2+3a\underset{0}{h}+\underset{0}{h^2}+2)}{\cancel{h}} = \boxed{f' = 3a^2+2}$$

Find $f'(x)$ for $f(x)$. Then find the slope of the graph of f at the points $(1,1)$ and $(4,2)$.

$f(x) = \sqrt{x}$

$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}\cdot\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} =$$

$$\lim_{h\to 0}\frac{\cancel{a}+\cancel{h}-\cancel{a}}{\cancel{h}(\sqrt{a+\underset{0}{h}}+\sqrt{a})} =$$

$$m = \frac{1}{\sqrt{a}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}} = m}$$

$$\boxed{\text{at }(1,1)\quad m = \frac{1}{2\sqrt{1}} = \frac{1}{2}}$$

$$\boxed{\text{at }(4,2)\quad m = \frac{1}{2\sqrt{4}} = \frac{1}{4}}$$

Find the derivative with respect to t for the function y.

$y = \frac{2}{t}$

common denom.

$$f' = \lim_{h\to 0}\frac{\frac{2}{a+h}\frac{(a)}{(a)} - \frac{2}{a}\frac{(a+h)}{(a+h)}}{h}$$

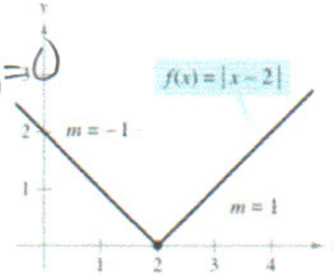
$$= \lim_{h\to 0}\frac{\frac{\cancel{2a}-\cancel{2a}-2\cancel{h}}{(a+h)(a)}}{h}\cdot\frac{1}{\cancel{h}}$$

$$\lim_{h\to 0}\frac{-2}{(a+\underset{0}{h})(a)} = \boxed{\frac{-2}{a^2} = f'}$$

The derivative of f at a is : $f'(x) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$ provided that the limit exists.
In order for the limit to exist, the one-sided limits must exist.

A function with a sharp turn

$f(x) = |x-2|$ $\quad f(a) = f(2) = |2-2| = 0$



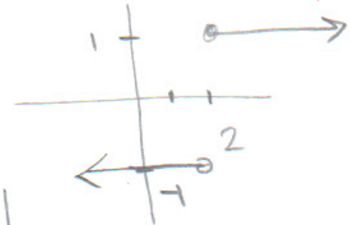
While $f(x)$ is continuous, is it differentiable at $x = 2$???

$$\lim_{x \to 2^-} \frac{|x-2| - 0}{x-2} =$$

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$$

from left

limits not equal

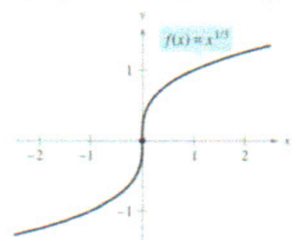


$$\lim_{x \to 2^+} \frac{|x-2| - 0}{x-2} = \lim_{x \to 2^+} \frac{|x-2|}{x-2} = 1$$

The limits from both sides do not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point (2,0).

Graph with vertical tangent line Look at $x = 0$

$f(x) = x^{1/3}$ $\quad f(0) = 0$



$$\lim_{x \to 0} \frac{x^{1/3} - 0}{x - 0} =$$

$$\lim_{x \to 0} \frac{x^{1/3}}{x} = \text{Bring down}$$

$$\lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

$x = .00001$ $f(x) = 2154.4$

$x = -.00001$ $f(x) = 2154.4$

we don't get an actual value for our limit

$\therefore$ the tangent line has no slope / vertical

Summary a function is not differentiable at a point at which its graph has a sharp turn or a vertical tangent line.

THEOREM 2.1 DIFFERENTIABILITY IMPLIES CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

The following statements summarize the relationship between continuity and differentiability.

1. **If a function is differentiable at x=c, then it is continuous at x=c. So, differentiability implies continuity.**
2. **It is possible for a function to be continuous at x=c and not be differentiable at x=c. So, continuity does not imply differentiability.**