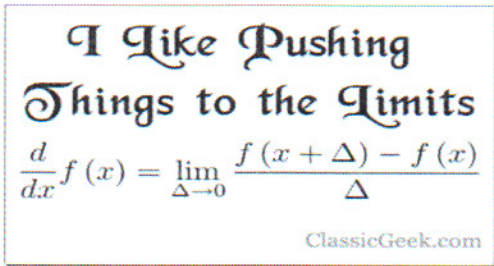
Calculus
Lesson 1.2-Finding Limits Graphically and Numerically
1.3 Finding Limits Analytically
Mrs. Snow, Instructor



From precalculus, we found several methods to get an idea of the behavior of the graph of f near an undefined value of x:

Estimating a Limit Numerically: When entering into calculator: Parentheses are a must!

$$f(x) = \lim_{x\to 0} \frac{(x)}{(\sqrt{x+1}-1)} = \underline{\underline{2}}$$

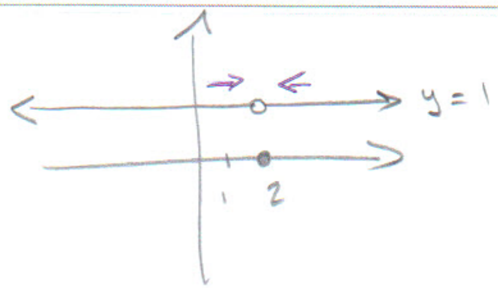
$(\quad)/(\quad)$

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.994	1.9994	1.9999	u	2.0000	2.0004	2.0049

Finding a Limit

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$$\lim_{x\to 2} f(x) = 1$$



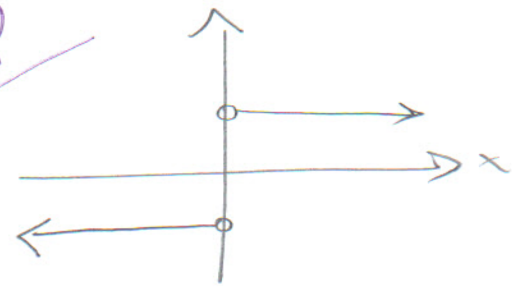
Approach from Left $x \to 2^-$
right $x \to 2^+$

Limits That Fail to Exist:

Behavior that differs from right to left:

$\lim\limits_{x \to 0} \frac{|x|}{x}$ = DNE

Jump



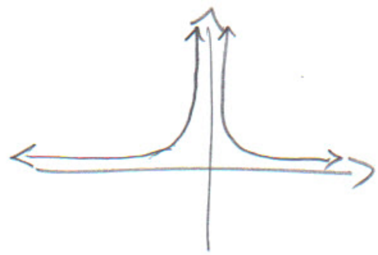
$x = 1$

$\lim\limits_{x \to 0^+} \frac{|1|}{1} = 1$

$\lim\limits_{x \to 0^-} \frac{|-1|}{-1} = -1$

Unbounded Behavior:

$\lim\limits_{x \to 0} \frac{1}{x^2}$ = DNE

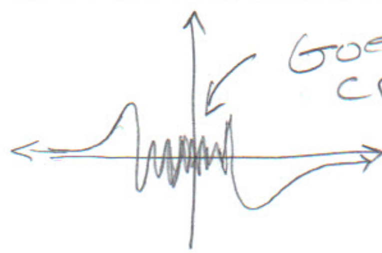


$\lim\limits_{x \to 0^+} \frac{1}{\left(\frac{1}{10}\right)^2} = 100$

$\lim\limits_{x \to 0^-} \frac{1}{\left(-\frac{1}{10}\right)^2} = 100$

Oscillating Behavior:

$\lim\limits_{x \to 0} \sin \frac{1}{x}$ = DNE



Goes crazy as we approach 0.

1.3 Finding Limits Analytically

We found that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however that the limit is precisely $f(c)$. In these cases, the limit may be evaluated by **direct substitution**:

THEOREM 1.1 SOME BASIC LIMITS

Let b and c be real numbers and let n be a positive integer.

1. $\lim\limits_{x \to c} b = b$ **2.** $\lim\limits_{x \to c} x = c$ **3.** $\lim\limits_{x \to c} x^n = c^n$

a. $\lim\limits_{x \to 2} 3 = 3$ *b.* $\lim\limits_{x \to -4} x = -4$ *c.* $\lim\limits_{x \to 2} x^2 = 2^2 = 4$

THEOREM 1.2 PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [b f(x)] = bL$ $\quad$ that is : $b \lim_{x\to c} f(x) = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

The Limit of a Polynomial:

$$\lim_{x\to 2}\left(4x^2+3\right) = 4(2^2)+3$$
$$= 4(4)+3$$
$$= 16+3 = \underline{\underline{19}}$$

The **direct substitution** property is valid for all polynomial and rational functions with nonzero denominators:

THEOREM 1.3 LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

The Limit of a Rational Function

$$\lim_{x\to 1}\frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = \underline{\underline{2}}$$

THEOREM 1.4 THE LIMIT OF A FUNCTION INVOLVING A RADICAL

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

THEOREM 1.5 THE LIMIT OF A COMPOSITE FUNCTION

If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

We see that the limits of many algebraic functions may be evaluated by direct substitution. The six basic trigonometric functions also exhibit a desirable quantity:

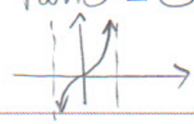
THEOREM 1.6 LIMITS OF TRIGONOMETRIC FUNCTIONS

Let c be a real number in the domain of the given trigonometric function.

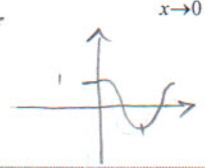
1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \cot x = \cot c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \csc x = \csc c$

New! Know!

$\lim_{x\to 0} \tan x =$ $\tan 0 = 0$



$\lim_{x\to \pi} x\cos x =$ $= \pi \cos \pi$ $= \pi(-1)$ $= -\pi$



$\lim_{x\to 0} \sin^2 x = \sin^2(0)$ $= 0$

A Strategy for Finding Limits

We have reviewed several types of functions whose limits may be evaluated by direct substitution. This knowledge and Theorem 1.7 may be used to develop a strategy for finding limits:

THEOREM 1.7 FUNCTIONS THAT AGREE AT ALL BUT ONE POINT

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Finding the limits of a Function: Factor

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = \lim_{x\to 1} x^2+x+1$$

$$= 1+1+1 = \underline{\underline{3}}$$

Dividing Out Technique: (factor numerator)

$$\lim_{x\to -3}\frac{x^2+x-6}{x+3} = \lim_{x\to -3}\frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}}$$

$$= \lim_{x\to -3}(x-2) = -3+2 = -5$$

Rationalizing Technique: conjugates : $(a+b)$ & $(a-b)$

$$\lim_{x\to 0}\frac{(\sqrt{x+1}-1)}{(x)}\cdot\frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{x+\cancel{1}-\cancel{1}}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x\to 0}\frac{\cancel{x}\ 1}{\cancel{x}(\sqrt{x+1}+1)}$$

$$= \frac{1}{\sqrt{1}+1} = \underline{\frac{1}{2}}$$

THEOREM 1.9 TWO SPECIAL TRIGONOMETRIC LIMITS

1. * $\lim_{x\to 0}\frac{\sin x}{x} = 1$ 2. $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$ Know !

A Limit Involving a Trigonometric Function:

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to 0}\frac{\sin x}{\cos x}\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0} * \frac{\sin x}{x}\left(\frac{1}{\cos x}\right) = \frac{1}{\cos 0} = \underline{1}$$

$$\lim_{x\to 0}\frac{\sin 4x}{x} = \lim_{x\to 0}\frac{4\sin x\cos x\cos 2x}{x}$$

$$= \lim_{x\to 0} 4\cos x\cos 2x$$

$$= 4\cos 0\cos 0 = \underline{4}$$

$\sin 4x = \sin 2(2x)$

$= 2\sin 2x\cos 2x$

$= 2(2)\sin x\cos x\cos 2x$

$= 4\sin x\cos x\cos 2x$

$\sin 2x = 2\sin x\cos x$