

Calculus
 Lesson 1.4: Continuity and One-Sided Limits
 Mrs. Snow, Instructor

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

LEFT-HANDED LIMIT (by a right-handed mathematician!)
 2007 @ Courtney Gleason

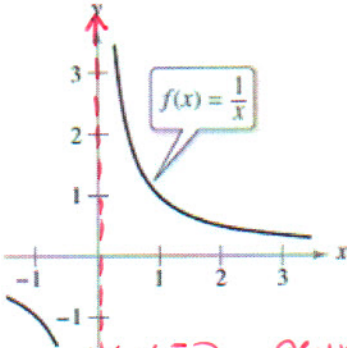
In mathematics the term, *continuous*, pretty much means the same as it has in everyday language. When a function f is continuous at $x = c$, it means that there is no interruption in the graph of f at c . The graph is smooth, that is there are no breaks, no holes, no jumps or gaps.

If a function f is not continuous at c , then f is said to have a **discontinuity** at c . Discontinuities fall into two categories: **removable** and **nonremovable**. In other words, can you remove the discontinuity or is it there no matter what.

Continuity of a Function: nonremovable-

$$f(x) = \frac{1}{x}$$

Domain: $\{x | x \in \mathbb{R}, x \neq 0\}$ \therefore **DISCONTINUOUS**



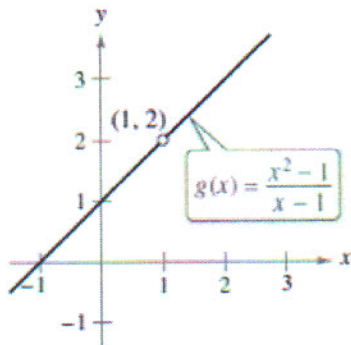
f has a nonremovable discontinuity at $x = 0$. In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

removable-

$$g(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1}$$

Hole at $x=1$

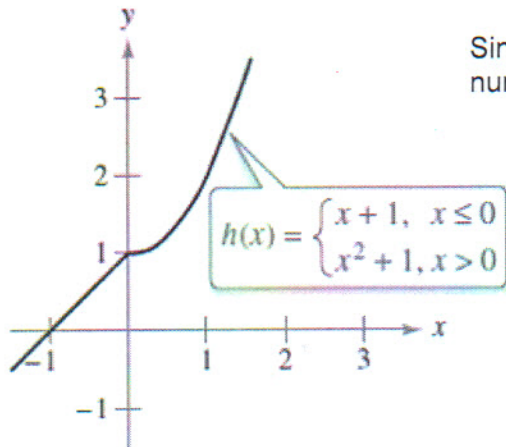
Domain: $\{x | x \in \mathbb{R}, x \neq 1\}$ \therefore **DISCONTINUOUS**



At $x = 1$, the function has a removable discontinuity. If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers

Continuous-

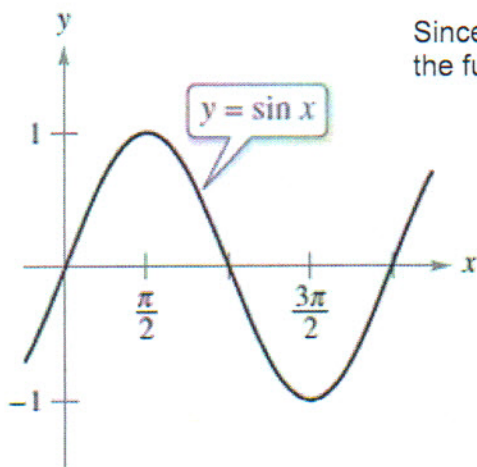
Domain: $\{x|x \in \mathbb{R}\} \therefore$ **CONTINUOUS**



Since the domain of the function is all real numbers, the function is continuous.

$y = \sin x$

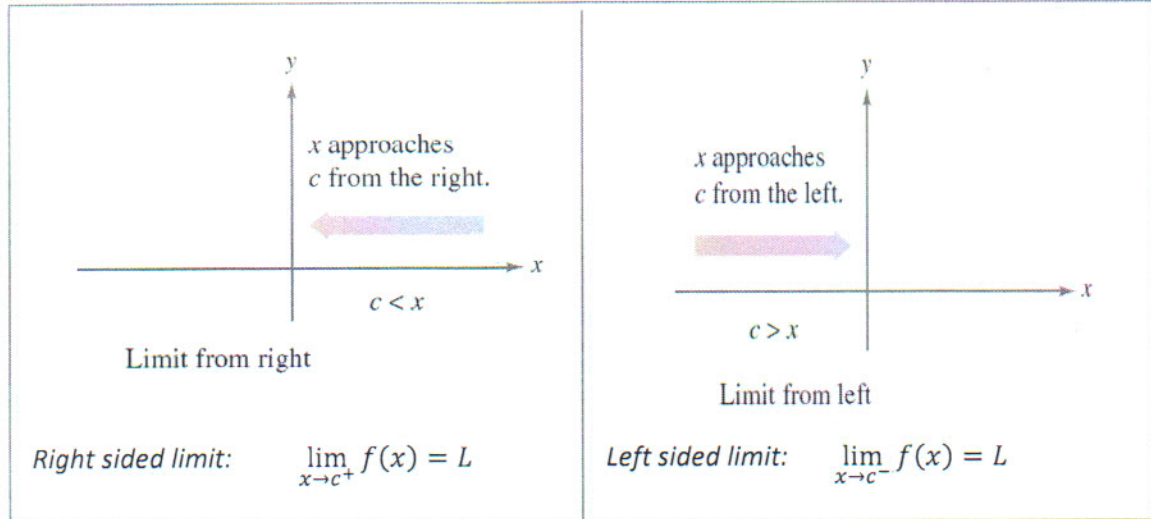
Domain: $\{x|x \in \mathbb{R}\} \therefore$ **CONTINUOUS**



Since the domain of the function is all real numbers, the function is continuous.

One-Sided Limits

From precalculus we learned that to verify the limit of a function we have to look at the **one-sided limits**. Another need for one-sided limits is so to understand continuity on a closed interval.



Find the limit of $f(x)$ as x approaches -2 from the right.

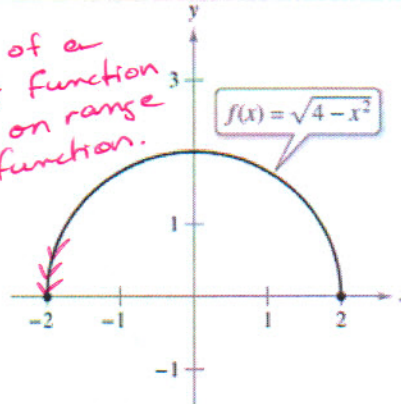
$$f(x) = \sqrt{4 - x^2}$$

SO:

$$\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0$$

As x approaches -2 from the right y gets closer to 0

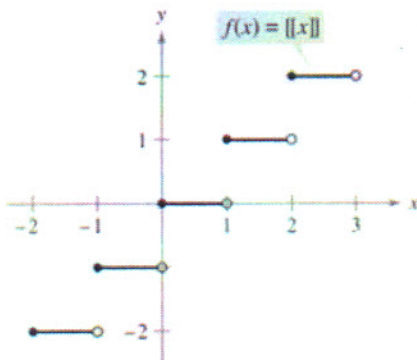
← equation of a circle - not function so restriction on range to make a function.



The Greatest Integer Function a.k.a. a step function: One-sided limits can be used to investigate the behavior of step functions.

$\lceil x \rceil =$ *greatest integer n such that $n \leq x$*

Find the limit of the greatest integer function as x approaches 0 from the left and the right.



$$\lim_{x \rightarrow 0^-} \lceil x \rceil = -1$$

$$\lim_{x \rightarrow 0^+} \lceil x \rceil = 0$$

What is happening at zero?

Discontinuity at zero
Where do we see discontinuities?

at any integer n .

When the limit from the left is not equal to the limit from the right, the limit *does not exist*.

THEOREM 1.10 THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

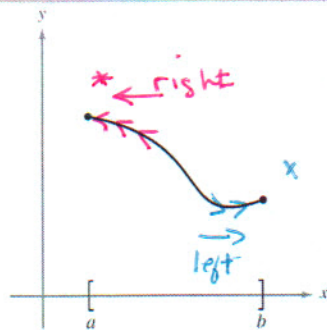
$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

DEFINITION OF CONTINUITY ON A CLOSED INTERVAL

A function f is **continuous on the closed interval** $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \rightarrow b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31). * x



Continuous function on a closed interval

Properties of Continuity:

THEOREM 1.11 PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c .

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, if $g(c) \neq 0$

Applying Properties of Continuity

$f(x) = x + \sin x$ $f(x) = 3 \tan x$ $f(x) = \frac{x^2 + 1}{\cos x}$

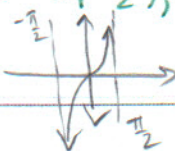
Both continuous
 $\therefore f(x)$ continuous

continuous

both continuous so f(x) continuous

$\dots \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \dots$

Hence, all are continuous.

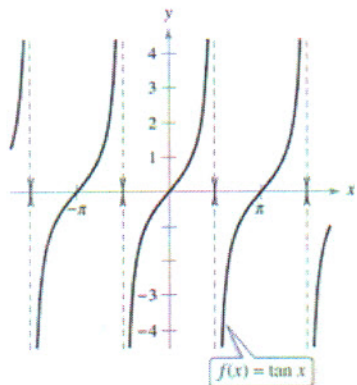


THEOREM 1.12 CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c .

Testing for Continuity

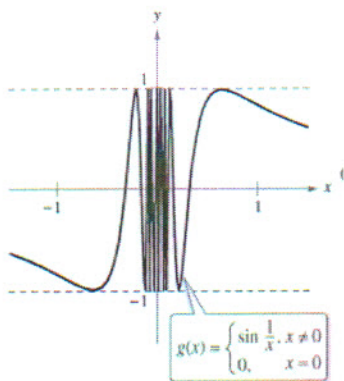
$f(x) = \tan x$



The tangent graph has asymptotes at all odd multiples of $\frac{\pi}{2}$. So it is continuous at all other intervals. Thus, we can say that $f(x) = \tan x$ is continuous on the open intervals:

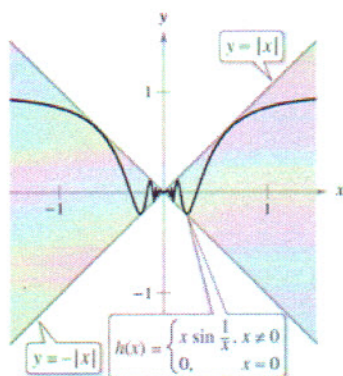
$\dots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \dots$

$g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



$\sin\left(\frac{1}{x}\right)$ is continuous everywhere except $x = 0$. At $x = 0$ the limit of $g(x)$ does not exist. So $g(x)$ is continuous on the intervals: $(-\infty, 0)$ and $(0, \infty)$

$h(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



This function is similar to the previous function except that the oscillations are damped by the factor x . So $h(x)$ is continuous on the entire real line.

THEOREM 1.13 INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

So, " c " exists, but we are not given a method to find " c ".

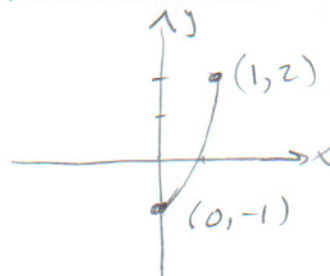
An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the following polynomial function has a zero in the interval $[0, 1]$. $[a, b]$

$$f(x) = x^3 + 2x - 1$$

$$f(a) = f(0) = 0^3 + 2(0) - 1 = -1 \quad \therefore (0, -1)$$

$$f(b) = f(1) = 1^3 + 2(1) - 1 = 2 \quad \therefore (1, 2)$$



Conclusion

So $f(x)$ is continuous over interval $[0, 1]$. Since $f(a) \neq f(b)$ have different signs, we know

that there is at least one zero over the closed interval $[0, 1]$.

In intermediate value thm, (transl.)

for a continuous function f , if x takes on all values between a & b then $f(x)$ must take on all values between $f(a)$ & $f(b)$.