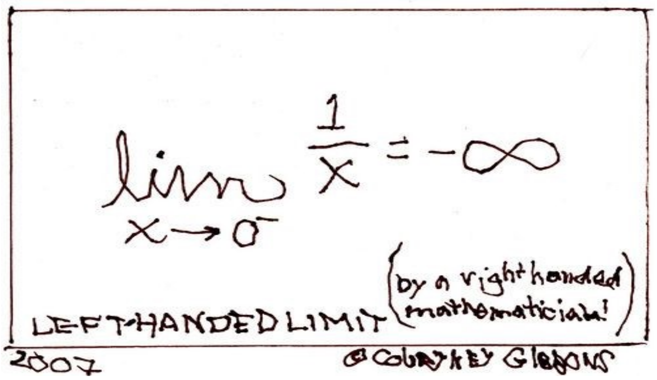
Calculus
Lesson 1.4: Continuity and One-Sided Limits
Mrs. Snow, Instructor



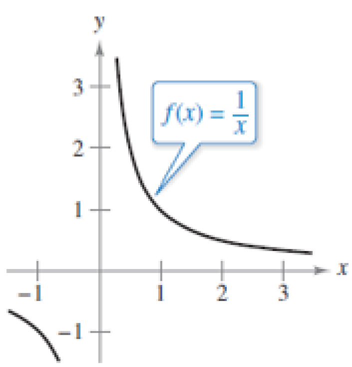
In mathematics the term, *continuous*, pretty much means the same as it has in everyday language. When a function f is continuous at $x = c$, it means that there is no interruption in the graph of f at c. The graph is smooth, that is there are no breaks, no holes, no jumps or gaps.

If a function f is not continuous at c, then f is said to have a **discontinuity** at c. Discontinuities fall into two categories: **removable and nonremovable.** In other words, can you remove the discontinuity or is it there no matter what.

Continuity of a Function: nonremovable-

$f(x) = \frac{1}{x}$

Domain: $\{x | x \in \mathbb{R}, x \neq 0\}$ $\therefore$ ***DISCONTINUOUS***

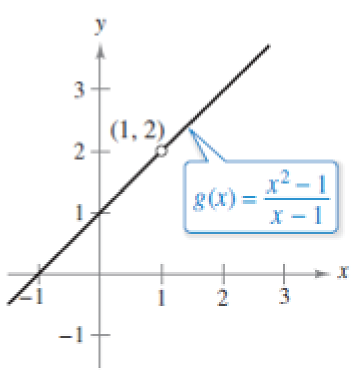


F has a <u>nonremovable discontinuity</u> at $x = 0$. In other words, there is no way to define $f(0)$ so as to make the function continuous at $x = 0$.

removable-

$g(x) = \frac{x^2 - 1}{x - 1}$

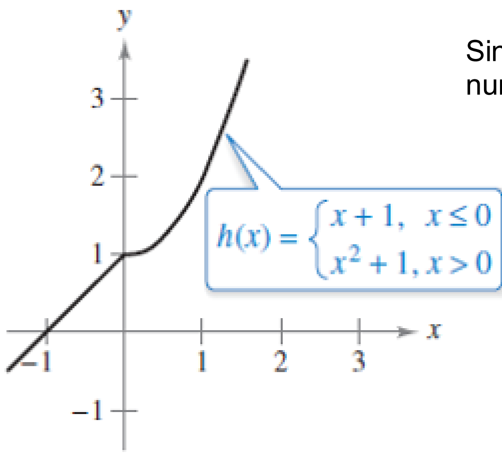
Domain: $\{x | x \in \mathbb{R}, x \neq 1\}$ $\therefore$ ***DISCONTINUOUS***



At $x = 1$, the function has a <u>removable discontinuity</u>. If $g(1)$ is defined as 2, the "newly defined" function is continuous for all real numbers

Continuous-

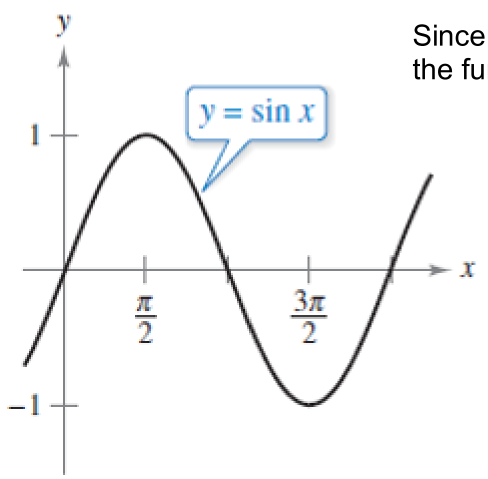
$\boldsymbol{Domain}$: $\{x|x \in \mathbb{R}\}$ $\therefore$ ***CONTINUOUS***



Since the domain of the function is all real numbers, the function is continuous.

$y = \sin x$

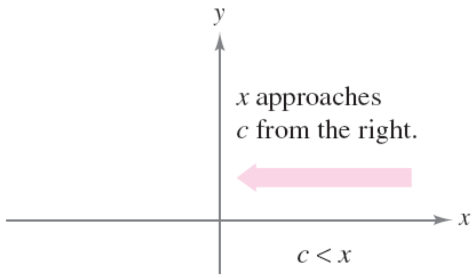
$\boldsymbol{Domain}$: $\{x|x \in \mathbb{R}\}$ $\therefore$ ***CONTINUOUS***



Since the domain of the function is all real numbers, the function is continuous.

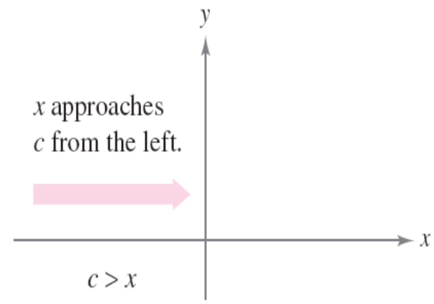
One-Sided Limits

From precalculus we learned that to verify the limit of a function we have to look at the **one-sided limits:**



Limit from right

Right sided limit: $\lim_{x \to c^+} f(x) = L$



Limit from left

Left sided limit: $\lim_{x \to c^-} f(x) = L$

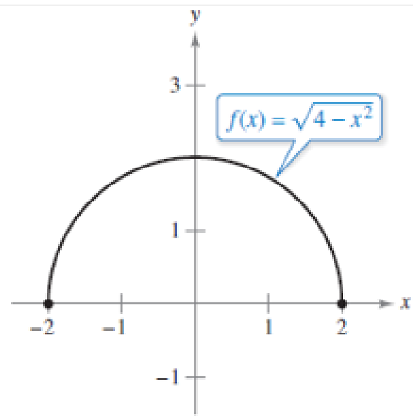
Find the limit of f(x) as x approaches -2 from the right.

$$f(x) = \sqrt{4 - x^2}$$

SO:

$$\lim_{x \to 2^+} \sqrt{4 - x^2} = 0$$

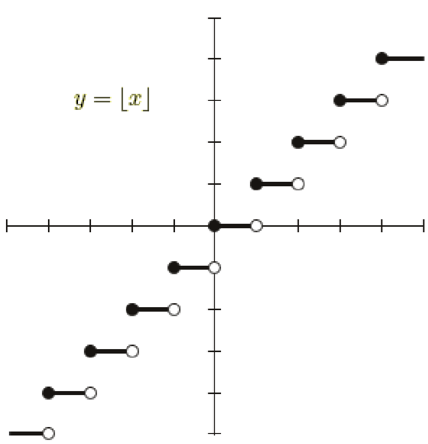
As x approaches -2 from the right y gets closer to 0



The Greatest Integer Function a.k.a. *a step function*

$[\![x]\!] = \boldsymbol{greatest\ integer\ n\ such\ that\ n \leq x}$

Find the limit of the greatest integer function as x approaches 0 from the left and the right.



$$\lim_{x \to 0^-} [[x]] = -1$$

$$\lim_{x \to 0^+} [[x]] = 0$$

When the limit from the left is not equal to the limit from the right, the limit *does not exist.*

THEOREM 1.10 THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

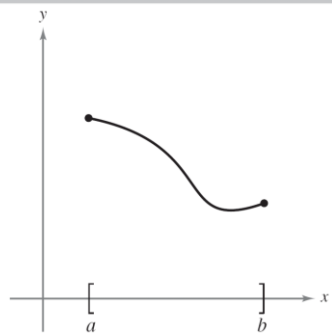
$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

DEFINITION OF CONTINUITY ON A CLOSED INTERVAL

A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b (see Figure 1.31).



Continuous function on a closed interval

Properties of Continuity:

THEOREM 1.11 PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, if $g(c) \neq 0$

Applying Properties of Continuity

$f(x) = x + \sin x \qquad f(x) = 3\tan x \qquad f(x) = \dfrac{x^2+1}{\cos x}$

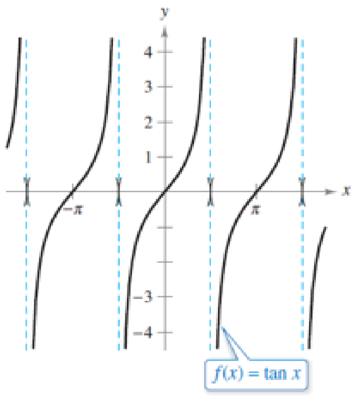
Hence, all are continuous.

THEOREM 1.12 CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

Testing for Continuity

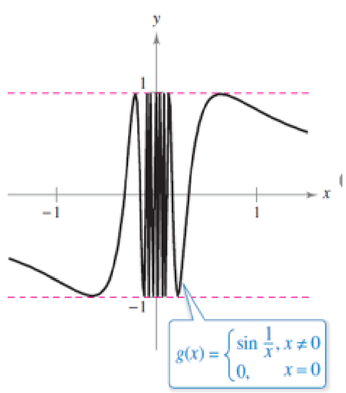
$f(x) = \tan x$



The tangent graph has asymptotes at all odd multiples of $\frac{\pi}{2}$. So it is continuous at all other intervals. Thus, we can say that $f(x) = \tan x$ is continuous on the open intervals:

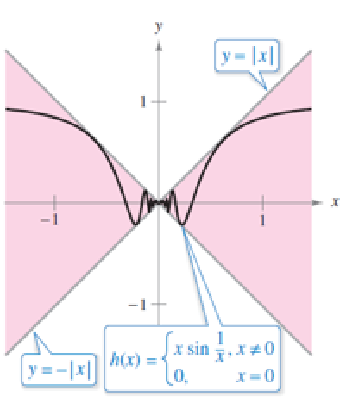
$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

$$g(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



$Sin\left(\frac{1}{x}\right)$ is continuous everywhere except $x = 0$. At $x = 0$ the limit of $g(x)$ does not exist. So $g(x)$ is continuous on the intervals: $(-\infty, 0)$ *and* $(0, \infty)$

$$h(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



This function is similar to the previous function except that the oscillations are damped by the factor x. So $h(x)$ is continuous on the entire real line.

THEOREM 1.13 INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

An Application of the Intermediate Value Theorem

Use the Intermediate Value Theorem to show that the following polynomial function has a zero in the interval $[0,1]$.

$$f(x) = x^3 + 2x - 1$$