Precalculus
Lesson 14.3: The Tangent Problem: The Derivative
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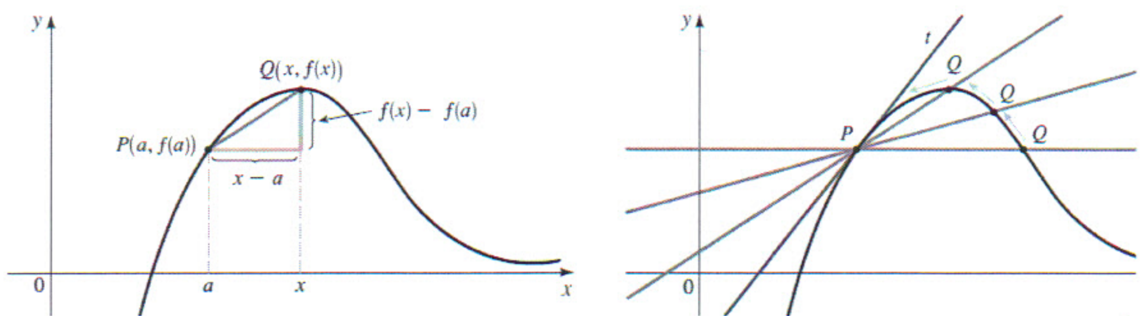
The slope of a line is used in application problems where the slope, also known as a rate of change, like a velocity. When the function is non-linear, the rate of change is not constant, so we can find the average velocity. What if we want (or maybe we don't want to but, have to .....) to find the exact velocity at an exact moment in time???? This is where the slope of a line tangent to a curve comes into play, and limits. Limits may be used to calculate the slope of a line that is tangent to a point on a curved graph. As a little foreshadowing, this tangent line problem and calculating its slope gave rise to the branch of calculus called *differential calculus* which dates back to the early to mid- 1600s!

If we think about our geometry definition of a tangent line, it is words to the effect of: A line that just touches a curve at one point, without cutting across it. In calculus we get the following:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



Find an equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point (3,1).

$f(x) = \frac{3}{x}$ Point $(a, f(a))$

$a = 3$

$f(a) = 1$

$$M = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to 3} \frac{\frac{3}{x} - 1}{x - 3}$$

$$= \lim_{x \to 3} \frac{\left(\frac{3}{x} - 1\right)}{(x - 3)}\left(\frac{x}{x}\right)$$

Clear out fraction - multiply by "1" $= \frac{x}{x}$

$$\lim_{x \to 3} \frac{3 - x}{(x - 3)(x)}$$

Almost but not quite: factor out -1

$$\lim_{x \to 3} \frac{-1\cancel{(x-3)}}{x\cancel{(x-3)}}$$

Direct Substitution

$$= \frac{-1}{3} = \underline{\frac{-1}{3} = M}$$

Equation: at (3,1)

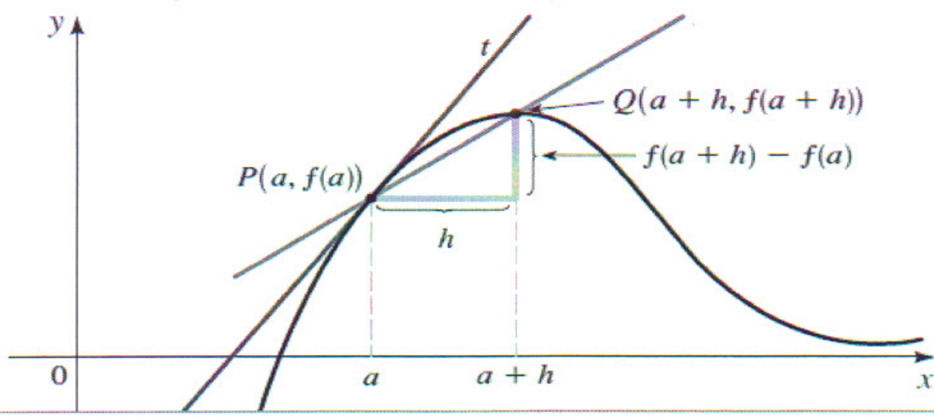
$y - 1 \overset{+1}{=} \frac{-1}{3}(x - 3) + 1$

$\boxed{y = -\frac{1}{3}x + 2}$

Another way to calculate slope is as follows:

$$\textit{average rate of change} = \frac{\textit{displacement}}{\textit{time}} = \frac{f(a+h) - f(a)}{h}$$

What is the slope of the line that contains the points P and Q?



$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Finding a Tangent Line

Find an equation of the tangent line to the curve $y = x^3 - 2x + 3$ at the point (1,2).

$1+h \quad 1+h$

$(a, f(a))$

$(1, f(1))$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^3 - 2(1+h) + 3 - 2}{h}$$

* Binomial theorem or Pascal's Δ

$$= \lim_{h \to 0} \frac{\cancel{1} + \cancel{3h} + 3h^2 + h^3 - \cancel{2} - \cancel{2h} + \cancel{3} - \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + h^3 + h}{h}$$

Should simplify such that "h" will factor out in next step

$$= \lim_{h \to 0} \frac{\cancel{h}(3h + h^2 + 1)}{\cancel{h}}$$

direct substitute

$= \underline{m = 1}$ $\therefore$ $y - 2 = 1(x - 1)$

$$\boxed{y = x + 1}$$

Ok, so we are looking at tangent lines and their respective slopes, we also know slope as *rate of change*. This is a very important concept in calculus and applications in science and engineering as many problems deal with motion and hence, the need to be able to determine not only the rate of change of a particle's movement, but the ***instantaneous rate of change*** of the movement of an object. Because this type of limit is so important it is given a special name and notation:

Definition of a Derivative

The derivative of a function is the slope of the tangent line…so…

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Finding a Derivative at a Point

(2+h)

Find the derivative of the function $f(x) = 5x^2 + 3x - 1$ at the number 2. $= a$

$f(a) = 5(4) + 3(2) - 1$
$= 20 + 6 - 1 = 25 = f(a)$

$f'(a) =$

$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} =$

$= \lim_{h \to 0} \frac{5(2+h)^2 + 3(2+h) - 1 - 25}{h}$

$= \lim_{h \to 0} \frac{5(4 + 4h + h^2) + 6 + 3h - 1 - 25}{h}$

$= \lim_{h \to 0} \frac{\cancel{20} + 20h + 5h^2 \cancel{+6} + 3h - \cancel{26}}{h}$

$= \lim_{h \to 0} \frac{23h + 5h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(23 + \cancel{5h}^{\,0})}{\cancel{h}} = 23$

So: $f'(a)$ = derivative = slope; the slope of a line through the point (2, 25) & tangent to $y = 5x^2 3x - 1$ is $23 = m$.

Let $f(x) = \sqrt{x}$ $\therefore$ $f(a) = \sqrt{a}$

Find the derivatives: $f'(a), f'(1), f'(4),$ and $f'(9)$

$f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$

$= \lim_{x \to a} \frac{\cancel{(x - a)}\ 1}{\cancel{(x - a)}(\sqrt{x} + \sqrt{a})}$

$= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}} = f(a)$

a

Now find $f'(1)$, $f'(4)$ & $f'(9)$

$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

Using the other Formulas:

problems from previous page

$f(x) = 5x^2 + 3x - 1$ at $a = 2$

$f(a) = 25$

$$f'(a) = \lim_{x \to 2} \frac{5x^2 + 3x - 1 - (25)}{x - 2}$$

$$= \lim_{x \to 2} \frac{5x^2 + 3x - 26}{x - 2}$$

$$= \lim_{x \to 2} \frac{(5x + 13)\cancel{(x-2)}}{\cancel{(x-2)}}$$

(hint: one of the factors should be equal to denominator, if math is correct)

$$= 2(5) + 13 = \underline{\underline{23}} =$$

$f(x) = \sqrt{x}$ $\quad f(a) = \sqrt{a}$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(\sqrt{a+h} - \sqrt{a})}{(h)} \cdot \frac{(\sqrt{a+h} + \sqrt{a})}{(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \to 0} \frac{\cancel{a} + h - \cancel{a}}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{a+h} + \sqrt{a})}$$

$$= \frac{1}{\sqrt{a} + \sqrt{a}} = \underline{\underline{\frac{1}{2\sqrt{a}}}} = f'(a)$$

Finding the instantaneous velocity in an application problem, the variables may be changed to represent what is going on in the problem.

If an object is dropped from a height of 3000 ft, its distance above the ground (in feet) after t seconds is given by $h(t) = 3000 - 16t^2$. Find the object's instantaneous velocity after 4 seconds.

$$h(t) = 3000 - 16t^2$$

$$t = 4 \text{ sec}: \quad h4 = 3000 - 16(16) = 2744 \text{ ft}$$

$$h' = \lim_{t \to 4} \frac{3000 - 16t^2 - 2744}{t-4}$$

$$= \lim_{t \to 4} = \frac{256 - 16t^2}{t-4} \quad \text{(factor)}$$

$$= \lim_{t \to 4} \frac{-16(t-16)}{t-4}$$

$$= \lim_{t \to 4} \frac{-16(t+4)\cancel{(t-4)}}{\cancel{(t-4)}}$$

$$\lim_{t \to 4} -16(t+4) = -16(8) = -128 \text{ ft/sec}$$

OR — $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$:

$$\lim_{h \to 0} \frac{3000 - 16(4+h)^2 - 2744}{h} =$$

$$\lim_{h \to 0} \frac{3000 - 16(16 + 8h + h^2) - 2744}{h}$$

$$\lim_{h \to 0} \frac{\cancel{3000} - \cancel{256} - 128h - 16h^2 - \cancel{2744}}{h}$$

$$\lim_{h \to 0} \frac{-128h - 16h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(-128 - \overset{0}{16h})}{\cancel{h}}$$

$$= -128 \text{ ft/sec}$$