Precalculus

Lesson 4.4: Properties of Rational Functions

Mrs. Snow, Instructor

When dealing with ratios of integers, they are identified as rational numbers. When we look at ratios of polynomials, we call them **rational functions.**

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Find the domain of the rational functions:

$R(x) = \frac{2x^3 - 4}{x + 5}$	$R(x) = \frac{1}{x^2 - 4}$
$R(x) = \frac{x^3}{x^2 + 1}$	$R(x) = \frac{x^2 - 1}{x - 1}$

Graph and analyze. What happens at $x = 0$? As x gets closer to 0?
What happens as $x \to \infty$?

$$R(x) = \frac{1}{x} \qquad\qquad H(x) = \frac{1}{x^2}$$

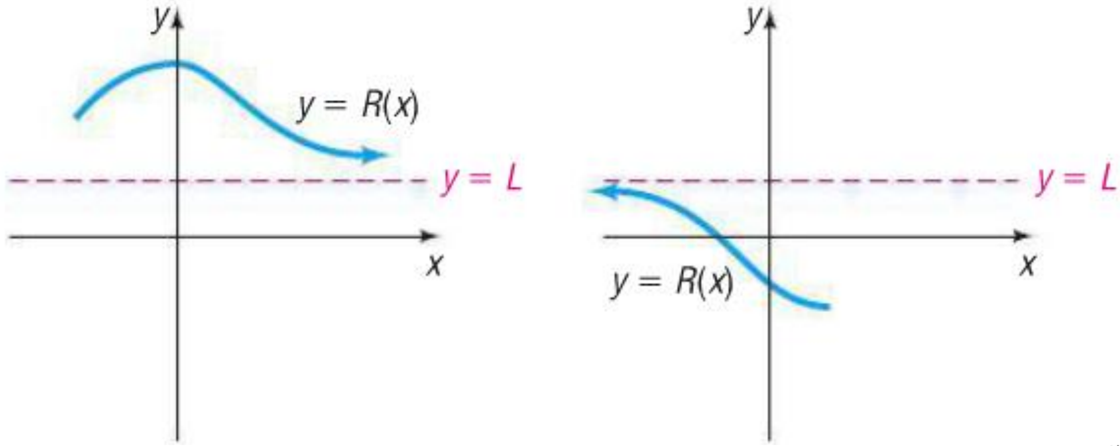
Graph the rational function using transformations:

$$R(x) = \frac{1}{(x-2)^2} + 1$$

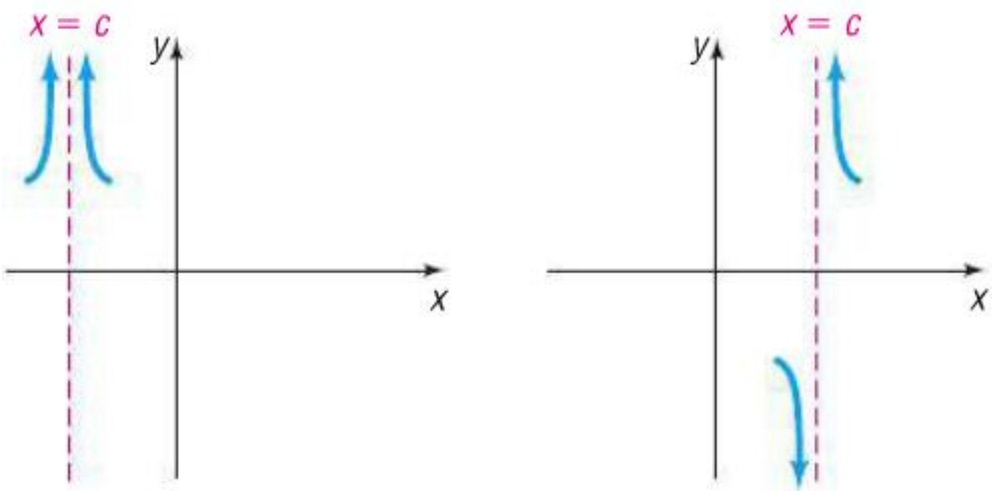
Asymptotes (see pg. 219 for more detail) *NOTE: Horizontal asymptotes may be intersected by the graph of a function!* *__The graph of a function will never intersect a vertical asymptote.__*

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.

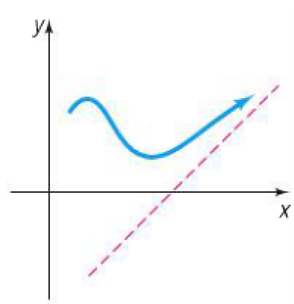
Horizontal asymptotes:



Vertical asymptotes:



There is also another type of asymptote, **OBLIQUE ASYMPTOTE**.



Vertical Asymptotes

- The values where the denominator goes to zero will be the vertical asymptotes; these are the domain restrictions and will graphically be seen as vertical asymptote(s).
- Factor denominator and set it equal to zero.

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \frac{5x^2}{3+x}$

(b) $R(x) = \frac{x}{x^2 - 4}$

(c) $H(x) = \frac{x^2}{x^2 + 1}$

(d) $G(x) = \frac{x^2 - 9}{x^2 + 4x - 21}$

Horizontal and Oblique Asymptotes

$$r(x)=\frac{a_n x^n + a_{n-1}x^{n-1}+...+a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1}+...+b_1 x + b_0}$$

Horizontal Asymtotes

1. Degree of denominator is bigger: $m > n$ horizontal asymptote at $y = 0$
   BOBO
2. Degree of numerator is bigger: $n > m$ no horizontal asymptote *BUT….**
   BOTN
3. Degrees of numerator and denominator are equal $n = m$: Exponents are the same – divide leading coefficients to find the horizontal asymptote
   EATS DC

***Oblique Asymptote**

When bigger on top, there will be an oblique asymptote. Divide the function. Quotient is the linear equation for the oblique asymptote.

$$\boldsymbol{r(x) = (ax+b) + \frac{r(x)}{q(x)}}$$

There are 2 possibilities that we will explore:

1. Numerator Degree bigger by 1. **Asymptote is Quotient**, **the line $\boldsymbol{y = ax + b}$**
2. Numerator Degree is bigger by more than 2.
   Quotient is a polynomial for degree 2 or higher.

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

For more detail see textbook pg 224

SUMMARY

Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).
2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.
   (a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
   (b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.
   (c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for very large values of $|x|$, the graph of R will behave like the graph of the quotient.

Note: The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote. ■