Precalculus
Lesson 9.5 The Dot Product
Mrs. Snow, Instructor

The concept of the dot product is used in calculus and in the applications of vectors in physics and engineering.

If $v = a_1 i + b_1 j = \langle a_1, b_1 \rangle$ and $w = a_2 i + b_2 j = \langle a_2, b_2 \rangle$ are vectors, then their dot product, denoted by $\boldsymbol{v} \cdot \boldsymbol{w}$, is defined by

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{a_1 a_2 + b_1 b_2}$$

say: "v dot w"

Given:

$$\langle 2, -3 \rangle \qquad \langle 5, 3 \rangle$$

$$v = 2i - 3j \quad \textit{and} \quad w = 5i + 3j$$

Find the following dot products:

a) $v \cdot w \;=\; (2)(5) + (-3)(3) = 10 - 9 = 1$

b) $w \cdot v \;=\; (5)(2) + (3)(-3) = -9 + 10 = 1$

c) $v \cdot v \;=\; (2)(2) + (-3)(-3) = 4 + 9 = 13$

d) $w \cdot w \;=\; (5)(5) + (3)(3) = 25 + 9 = 34$

e) $\|v\| \;=\; \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

f) $\|w\| \;=\; \sqrt{5^2 + 3^2} = \sqrt{25+9} = \sqrt{34}$

The following properties of the Dot Product are useful in solving problems involving the Dot Product:

$$u \cdot v = v \cdot u$$

$$(au) \cdot v = a(u \cdot v) = u \cdot (av)$$

$$u \cdot (v + w) = u \cdot v + u \cdot w$$

$$v \cdot v = \|v\|^2$$

$$0 \cdot v = 0$$

The Dot Product Theorem

If we have u and v be vectors with initial points at the origin, the angle θ that is between u and v is $0 < \theta < \pi$.

$$\boldsymbol{u \cdot v = \|u\|\|v\| cos\theta}$$

$$\boldsymbol{cos\theta = \frac{u \cdot v}{\|u\|\|v\|}}$$

Find the angle θ between $u = 4i - 3j \quad and \quad v = 2i + 5j$

$u \cdot v = (4)(2) + (-3)(5)$
$= 8 - 15$
$= -7$

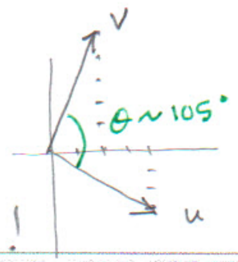
$\|u\| = \sqrt{16 + 9}$
$= \sqrt{25}$
$\|u\| = 5$

$\|v\| = \sqrt{4 + 25}$
$= \sqrt{29}$

$\cos\theta = = \dfrac{-7}{5\sqrt{29}}$

$\cos\theta = \sim .26$

$\theta \approx 105°$



*check w/ rough sketch!

Orthogonal Vectors (a.k.a. perpendicular)

Two vectors v and w are orthogonal, a.k.a. perpendicular, if and only if:

$$v \cdot w = 0$$

Determine whether the vectors in each pair are perpendicular

$$v = 2i - j \quad and \; w = 3i + 6j$$

$v \cdot w = 0$?

$(2)(3) + (-1)(6)$
$6 - 6 = 0$

yes orthogonal

Work (Dot product application)

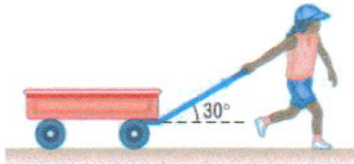
The work W done by a force F in moving along a vector D is

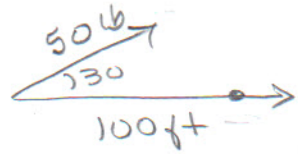
$$W = F \cdot D.$$

English units of force is pounds (lbs.)

When the force acting on the object is at an angle, remember to put into component form.

A girl is pulling a wagon with a force of 50 pounds. How much work is done in moving the wagon 100 feet if the handle makes an angle of 30° with the ground?





$W = F \cdot D$

$$F = 50\cos 30 i + 50 \sin 30 j$$

$$F = 50\left(\frac{\sqrt{3}}{2}\right)i + 50\left(\frac{1}{2}\right)j$$

$$F = 25\sqrt{3}\, i + 25\, j$$

$$D = 100 i + 0 j$$

$$W = F \cdot D = (25\sqrt{3})(100) + (25)(0)$$

$$W = 2500\sqrt{3}$$

* So : Direction of movement is horizontal

The only component of force involved in the work on the wagon is the horizontal component.