Precalculus
Lesson 9.4: Vectors
Mrs. Snow, Instructor

Many concepts in science involve applications of mathematics that measure certain quantities by their magnitude like length, mass, area, temperature, or energy. Only one number is needed to describe a length of 7 inches or 5°C for example. This single quantity is called **scalar.**

There are, however, many applications that involve not only the *magnitude* of an object but also, the *direction* of the displacement.
vector: a quantity that has both magnitude and direction. For example, the flight pattern of a plane, has both *speed (magnitude)* and *direction* of travel. Velocity, acceleration, and force are described by both magnitude and direction and are known as vectors.

(c) Directed line segment $\overrightarrow{PQ}$

P is the initial point
Q is the terminal point

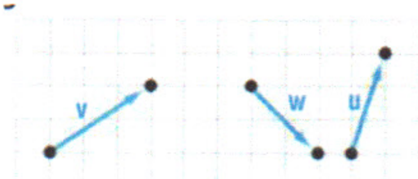
All vectors have two things:

Direction – follow the arrow.

Magnitude – the length of the vector.

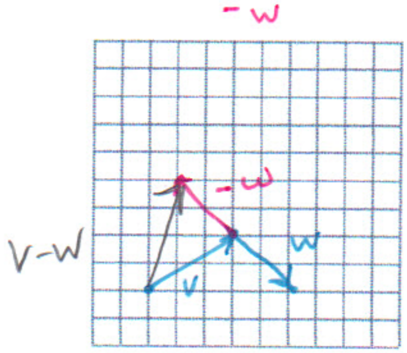
Graphing Vectors

Use the vector to graph each of the following vectors:

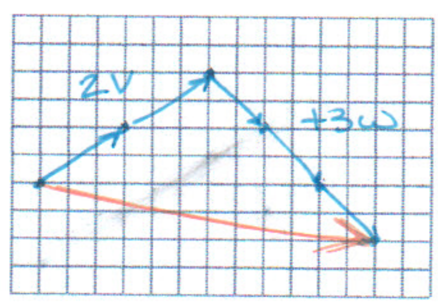


$v - w$

- w



$2v + 3w$



2v+3w

Find a Position Vector

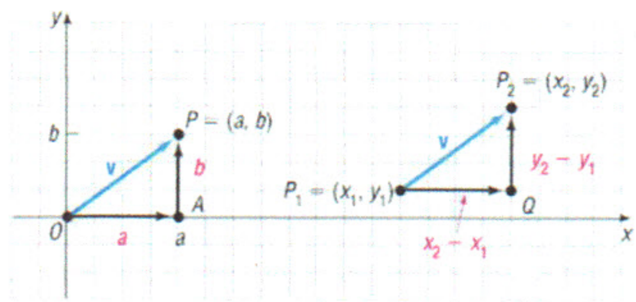
If we locate a vector in a coordinate plane we can describe it analytically by writing it in components.
Vector **v**, may be described with initial point P_1 (x_1, y_1) and terminal point P_2 (x_2, y_2), therefore:

$$v = \langle x_2 - x_1, \quad y_2 - y_1 \rangle$$

$$v = \langle a, b \rangle$$

This vector may be called the **position vector or component form**

$\mathbf{v} = \langle a, b \rangle = \langle x_2 - x_1, y_2 - y_1 \rangle$



Find the position vector **v** with initial point $(-1, 2)$ and terminal point $(4, 6)$.

$x_1 \; y_1$ $\qquad x_2 \; y_2$

$V = \langle 4 - (-1), \; 6 - 2 \rangle$

$= \langle 5, 4 \rangle$

Vectors in terms of *I and j*

A vector of length **1** is called a **unit vector**. The vector **w** $\langle \frac{3}{5}, \frac{4}{5} \rangle$ is an example of a **unit vector**.
We have two special unit vectors **i** and **j.**

"I" is a unit vector in the x-direction and **"j"** is a unit vector in the y-direction. Any vector in the x-direction can be written as a scalar multiple of **i** and any vector in the y-direction can be written as a scalar multiple of **j.** They are defined as:

$$\boldsymbol{i} = \langle \mathbf{1}, \mathbf{0} \rangle \;\; \boldsymbol{and} \;\; \boldsymbol{j} = \langle \mathbf{0}, \mathbf{1} \rangle, \; where \| \, i \| = \sqrt{1^2 + 0^2} \; and \; \|j\| = \sqrt{0^2 + 1^2}.$$

Any vector may be express in terms of **i** and **j.**

Algebraic Operations

Vectors may be added, subtracted, or have scalar multiplication. Pretty straight forward:

Let $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} = \langle a_2, b_2 \rangle$ be two vectors, and let α be a scalar. Then

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} = \langle a_1 + a_2, b_1 + b_2 \rangle \quad (2)$$

$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j} = \langle a_1 - a_2, b_1 - b_2 \rangle \quad (3)$$

$$\alpha\mathbf{v} = (\alpha a_1)\mathbf{i} + (\alpha b_1)\mathbf{j} = \langle \alpha a_1, \alpha b_1 \rangle \quad (4)$$

$$\|\mathbf{v}\| = \sqrt{a_1^2 + b_1^2} \quad (5)$$

$\|v\|$ = magnitude

$$If \;\; v = 2i + 3j = \langle 2, 3, \rangle \; and \; w = 3i - 4j = \langle 3, -4 \rangle,$$

$find$: $a)\ v + w$, $b)\ v - w$, $c)\ 3v$, $d)\ 2v - 3w$, and $\|v\|$

a) $v + w = 2i + 3j + 3i - 4j = (2+3)i + (3-4)j = 5i - j = \langle 5, -1 \rangle$

b) $v - w = 2i + 3j - (\overset{-}{3i}\overset{+}{-4j}) = (2-3)i + (3+4)j = -i + 7j = \langle -1, 7 \rangle$

c) $3v = 3(2i + 3j) = 6i + 9j = \langle 6, 9 \rangle$

d) $2v - 3w = 2(2i + 3j) - 3(3i - 4j) = 4i + 6j - 9i + 12j$

$= -5i + 18j = \langle -5, 18 \rangle$

e) $\|v\| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

A vector that represents speed and velocity of an object is called a **velocity vector**. A vector describing a force represents the direction and amount of force acting upon an object and is called a **force vector.**

Find a Vector from its Direction and Magnitude

Given the magnitude $\|v\|$ of a nonzero vector **v** and the **direction angle** α, $0° < \alpha < 360°$, between **v and i**, then:

$$v = \|v\|(\cos\alpha \boldsymbol{i} + \sin\alpha \boldsymbol{j})$$

Writing a Vector When Its Magnitude and Direction Are Given

A ball is thrown with an initial speed of 25 mph in a direction that makes an angle of 30° with the positive x-axis. Express the velocity vector v in terms of I and j. What is the initial speed in the horizontal direction? What is the initial speed in the vertical direction?

$V = \|V\| \cos\alpha i + \sin\alpha j$ 25 mph is magnitude

$V = 25(\cos 30 i + \sin 30 j)$ 30° is α

$= 25\left(\frac{\sqrt{3}}{2} i + \frac{1}{2} j\right)$

$= \frac{25\sqrt{3}}{2} i + \frac{25}{2} j$

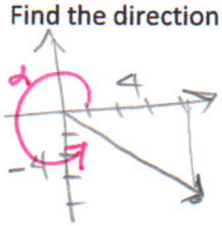
horizontal vertical

Initial speed in horizontal direction $= \frac{25\sqrt{3}}{2} \approx 21.65$ mph

In vertical direction $= \frac{25}{2} = 12.5$ mph

Finding the Direction Angle of a Vector

Find the direction angle α for $v = 4i - 4j$ (where is vector going?)



$0 < \alpha < 360°$

$\tan\alpha = \frac{-4}{4} = -1$ $\alpha = 45$ this is reference ∡!!

$\alpha = 315°$

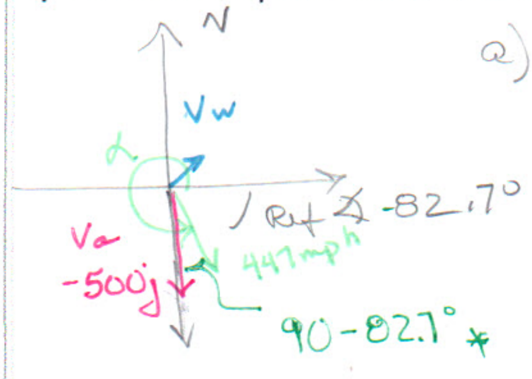
Finding the Actual Speed and Direction of an Aircraft

A Boeing 737 aircraft maintains a constant airspeed of 500 mph headed due south. The jet stream is 80 mph in the northeasterly direction. — Implies N 45° E

a) Express the velocity v_a of the 737 relative to the air and velocity v_w of the jet stream in terms of I and j.

b) Find the velocity of the 737 relative to the ground.

c) Find the actual speed and direction of the 737 relative to the ground.



a) $V_a = -500j$

$V_w = 80(\cos 45 i + \sin 45 j)$

$= 80\frac{\sqrt{2}}{2}i + 80\frac{\sqrt{2}}{2}j$

$V_w = 40\sqrt{2}i + 40\sqrt{2}j$

(b) V_g = Sum of $V_a + V_w$

$V_a = \quad\quad -500j$

$+V_w = 40\sqrt{2}i + 40\sqrt{2}j =$

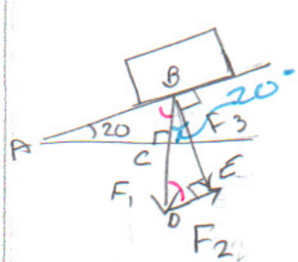
$V_g = 40\sqrt{2}i + (40\sqrt{2} - 500)j = V_g$

(c) $\|v_g\| = \sqrt{(40\sqrt{2})^2 + (40\sqrt{2}-500)^2} \approx 447$ mph

$\alpha \quad \tan\alpha = \frac{40\sqrt{2}-500}{40\sqrt{2}} \approx -82.7°$ Ref * S 7.3° E

Finding the Weight of a Piano

Two movers require a magnitude of force of 300 pounds to push a piano up a ramp inclined at an angle 20° from the horizontal. How much does the piano weigh?



3 forces F_1 force of gravity a.k.a weight

F_2 force to move piano = 300

F_3 force of piano against ramp

$\triangle ABC$ similar to $BDE \Rightarrow \angle EBD = 20°$

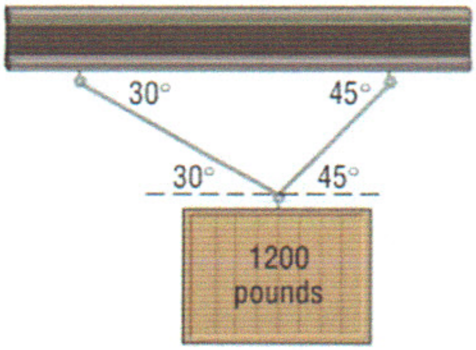
$\sin 20 = \frac{\|F_2\|}{\|F_1\|} = \frac{300}{\|F_1\|} \Rightarrow \|F_1\| = \frac{300}{\sin 20°}$

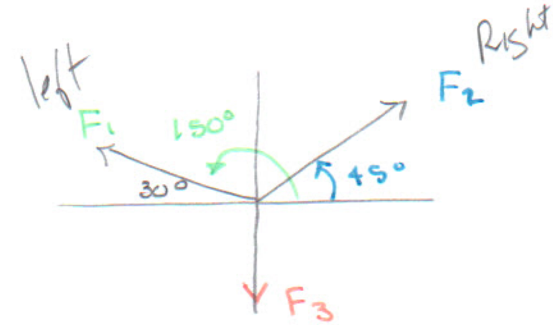
$= 877$ lb

weight of piano

An Object in Static Equilibrium: the object is at rest and the sum of all forces acting on the object is zero, a.k.a. the resultant force is zero.

A box of supplies that weighs 1200 pounds is suspended by two cables attached to the ceiling. What are the tensions in the two cables?





$$F_1 + F_2 + F_3 = 0$$ = Static Equilibrium

$$F_1 = \|F_1\|(\cos 150 i + \sin 150 j)$$
$$= \|F_1\|\left(-\frac{\sqrt{3}}{2} i + \frac{1}{2} j\right)$$
$$= -\frac{\sqrt{3}}{2}\|F_1\| i + \frac{1}{2}\|F_1\| j$$

$$F_2 = \|F_2\|(\cos 45 i + \sin 45 j) \qquad F_3 = -1200 j$$
$$= \frac{\sqrt{2}}{2}\|F_2\| i + \frac{\sqrt{2}}{2}\|F_2\| j$$

Horizontal

$$-\frac{\sqrt{3}}{2}\|F_1\| + \frac{\sqrt{2}}{2}\|F_2\| = 0$$

$$\frac{\sqrt{2}}{2}\|F_2\| = \frac{\sqrt{3}}{2}\|F_1\|$$

$$\|F_2\| = \frac{\cancel{2}}{\sqrt{2}} \frac{\sqrt{3}}{\cancel{2}}\|F_1\|$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\|F_1\|$$

$$\|F_2\| = \frac{\sqrt{3}}{\sqrt{2}}(878.5)$$

$$\|F_2\| = 1075.9 \text{ lb}$$

Right cable tension 1075.9 lb

Vertical:

$$\frac{1}{2}\|F_1\| + \frac{\sqrt{2}}{2}\|F_2\| - 1200 = 0$$

$$\frac{1}{2}\|F_1\| + \frac{\cancel{\sqrt{2}}}{2}\frac{\sqrt{3}}{\cancel{\sqrt{2}}}\|F_1\| - 1200 = 0$$

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\|F_1\| - 1200 = 0$$

$$\left(\frac{1+\sqrt{3}}{2}\right)\|F_1\| = 1200$$

$$\|F_1\| = 1200\left(\frac{2}{1+\sqrt{3}}\right)$$

$$\|F_1\| \approx 878.5 \text{ lb}$$

Left cable tension 878.5 lb