Precalculus
Lesson 9.2 Graphs of Polar Equations
Mrs. Snow, Instructor

To plot points with polar coordinates, it is convenient to use a polar grid. It is sort of like the unit circle superimposed with graph paper, like below:

Special graphs:
θ = constant – graphs a line at angle θ
r=constant – graphs a circle of radius r

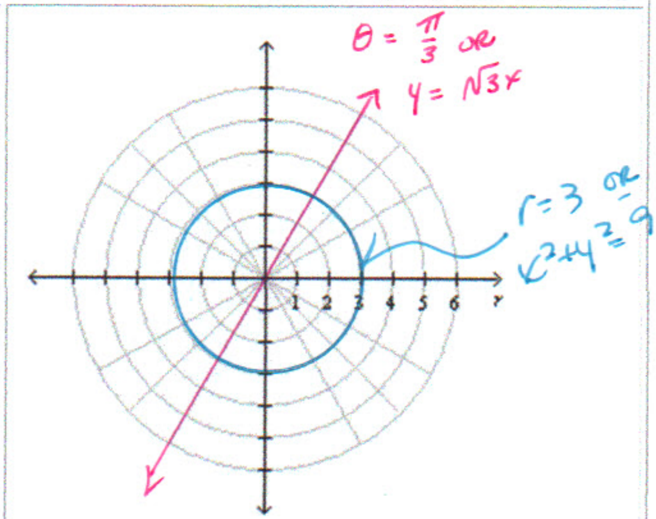
Sketch the graph of the equation and express the equation in rectangular coordinates:

$\theta = \frac{\pi}{3}$

$\tan\theta = \tan\frac{\pi}{3}$
$\frac{y}{x} = \sqrt{3}$
$y = \sqrt{3}x$
straight line

$r = 3$

$r^2 = 9$
$x^2 + y^2 = 9$
circle $r = 3$



Graphing a Polar Equation of a Line:

Some equations can easily be expressed in rectangular coordinates. If this is the case then convert to rectangular coordinates.

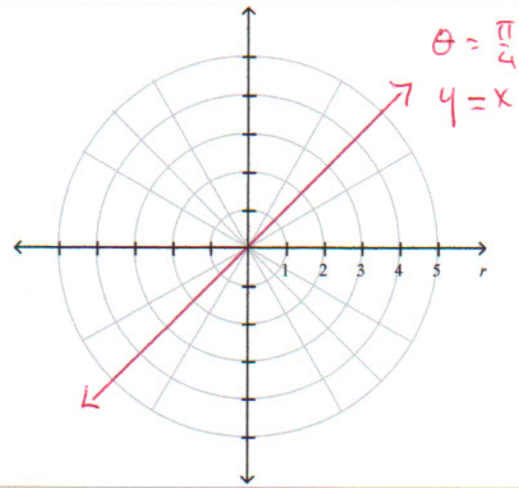
Identify and graph the equation

$$\theta = \frac{\pi}{4}$$

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1$$

$$y = x$$

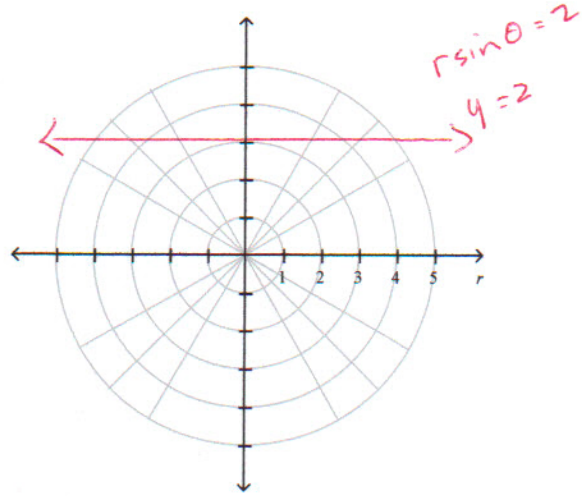


Identify and graph the equation

$$\underbrace{r\sin\theta} = 2$$

$$y = 2$$

Horizontal line

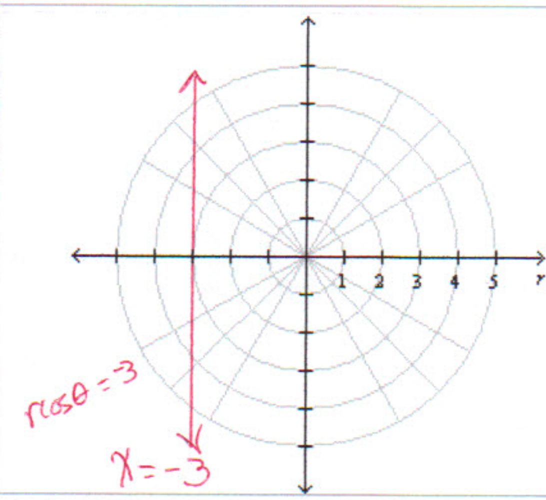


Identify and graph the equation

$$\underbrace{r\cos\theta} = -3$$

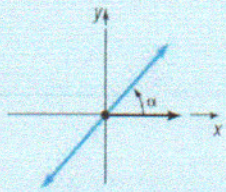
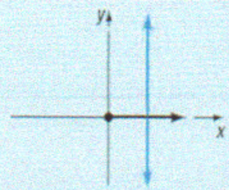
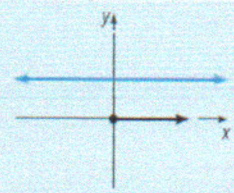
$$x = -3$$

Vertical line



In summary the equations in the forms below will graph as lines, note the forms for horizontal and vertical lines. Textbook (pg. 580:

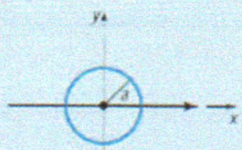
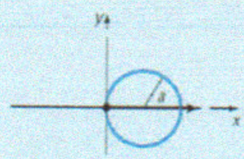
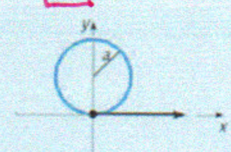
Lines

Description	Line passing through the pole making an angle α with the polar axis	Vertical line	Horizontal line
Rectangular equation	$y = (\tan\alpha)x$	$x = a$	$y = b$
Polar equation	$\theta = \alpha$	$r\cos\theta = a$	$r\sin\theta = b$
Typical graph			

$x = a$ $\qquad$ $y = b$

Identifying and Graphing a Polar Equation of a Circle (pg. 581):

Circles

Description	Center at the pole, radius a	Passing through the pole, tangent to the line $\theta = \frac{\pi}{2}$, center on the polar axis, radius a	Passing through the pole, tangent to the polar axis, center on the line $\theta = \frac{\pi}{2}$, radius a
Rectangular equation	$x^2 + y^2 = a^2$, $a > 0$	$x^2 + y^2 = \pm 2ax$, $a > 0$	$x^2 + y^2 = \pm 2ay$, $a > 0$
Polar equation	$r = a$, $a > 0$	$r = \pm 2a\cos\theta$, $a > 0$	$r = \boxed{\pm 2a}\sin\theta$, $a > 0$
Typical graph			

$2a$ = coefficient $\Rightarrow$ a = radius of circle

Sketch the polar equation (transform the equation into its rectangular form)

on $\frac{\pi}{2}$ axis $\quad r \cdot r = 4\sin\theta \cdot r$

$2a = 4$
$a = 2$
(radius)

$r^2 = 4r\sin\theta$

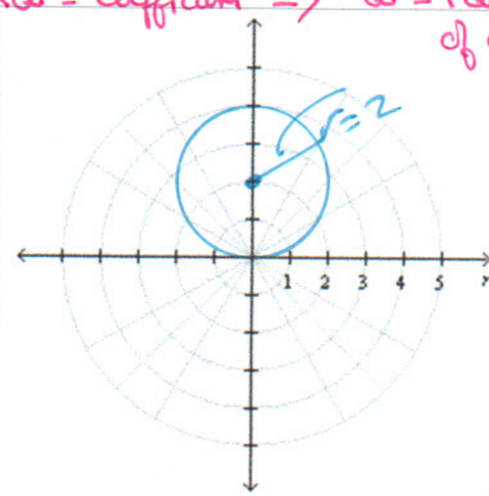
$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y + 4 = 0 + 4$

$(\frac{1}{2})(-4)$

$x^2 + (y-2)^2 = 4$

circle center $(0,2)$
$r = 2$



Sketch the polar equation

$$r = -2\cos\theta$$

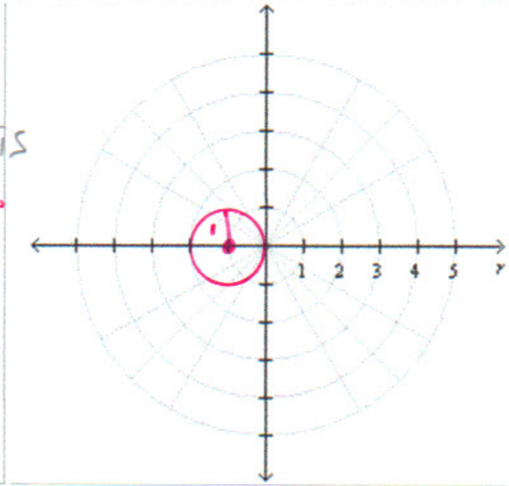
cosine along polar axis

Negative - polar axis

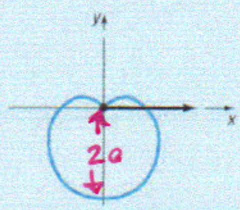
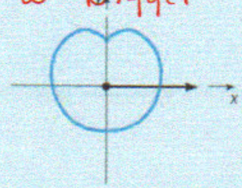
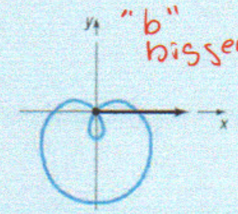
$2a = 2$

$a = -1$

radius $= 1$



Other Equations (pg. 581)

Other Equations			
Name	Cardioid	Limaçon without inner loop	Limaçon with inner loop
Polar equations	$r = a \pm a\cos\theta,\ a > 0$ $r = a \pm a\sin\theta,\ a > 0$	$r = a \pm b\cos\theta,\ 0 < b < a$ $r = a \pm b\sin\theta,\ 0 < b < a$	$r = a \pm b\cos\theta,\ 0 < a < b$ $r = a \pm b\sin\theta,\ 0 < a < b$
Typical graph			

addend & coefficient same #

2a

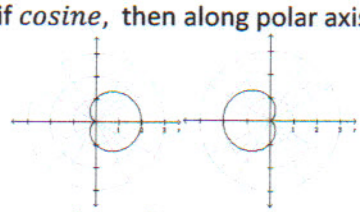
"a" bigger

"b" bigger

Limaçon: different values for addend & coefficient

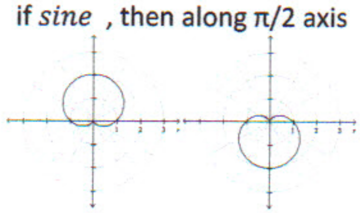
$a > 0$, *distance on axis is 2a*	if *cosine*: along polar axis if *sine*: along $\frac{\pi}{2}$ axis

if *cosine*, then along polar axis



$r = 1 + \cos\theta$ $\qquad$ $r = 1 - \cos\theta$

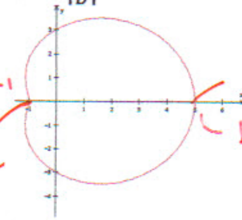
if *sine* , then along π/2 axis



$r = 1 + \sin\theta$ $\qquad$ $r = 1 - \sin\theta$

a. Limacon without inner loop if :

$1 < \left|\frac{a}{b}\right| < 2 \quad \rightarrow a > b$ $\qquad$ "a" bigger



b−a

$2-3=-1$

b−a

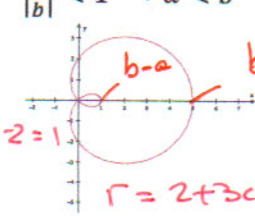
a+b

b+a=5

$r = 3 + 2\cos\theta$

b. Limacon has an inner loop if:

$\left|\frac{a}{b}\right| < 1 \quad \rightarrow a < b$ $\qquad$ "b" bigger



b−a

b+a

$2+3=5$

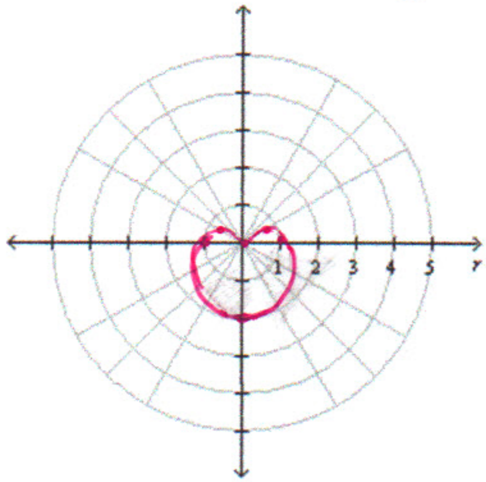
$3-2=1$

$r = 2+3\cos\theta$

Cardioid – heart shaped (pg. 581)

graph $r = 1 - \sin\theta$

$a = 1$; length of cardioid= $2a=2$ *equation is sine so along the* $\theta = \frac{\pi}{2}$ *axis and* $-\sin\theta$ *means … …* along $-\frac{\pi}{2}$ axis



Whenever you cannot remember how to graph the polar equation, you can always graph a period of the trig function from $0 \le \theta < 2\pi$ *and transfer the data over to a polar graph. Don't rely on memorizing an equation and associated graph shape, you will want a backup method!!*
Table of values *(use values for theta that yield friendly values for r):*

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	-1	0
$r = 1 - \sin\theta$	1	$\frac{1}{2}$	1-1 0	$\frac{1}{2}$	1	1-(-1) 2	1

Graphing a limaçon without an inner loop

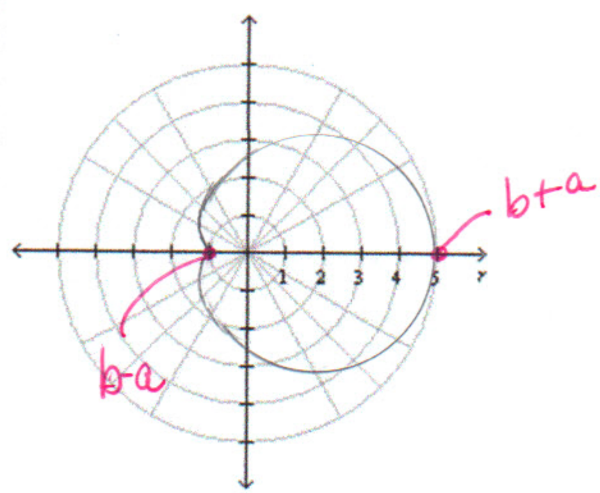
Sketch the graph of the equation
$r = 3 + 2\cos\theta$

$a = 3$ a bigger
$b = 2$ $\therefore$ no loop

$\cos\theta$ along polar axis

dimple at
$b - a = 2 - 3 = -1$

end at $b + a = 2 + 3$
$= 5$



Graphing a limaçon with an inner loop

Graphing a limaçon with an inner loop

$$r = 1 + 2\cos\theta$$

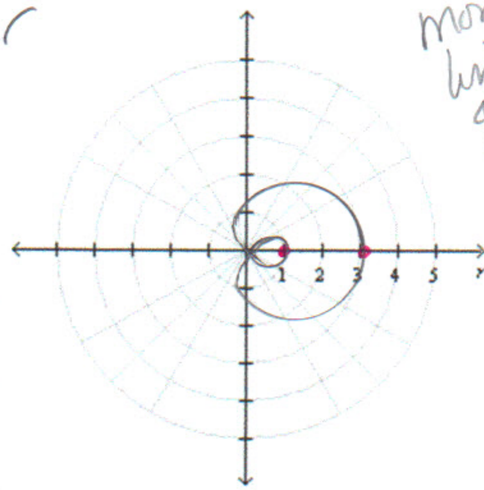
$a = 1$ "b" bigger; inner loop

$b = 2$

cosine along polar axis

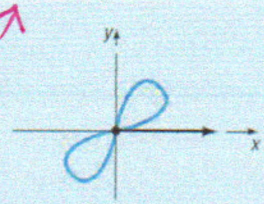
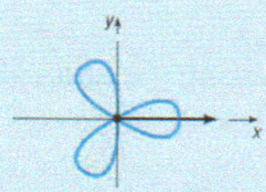
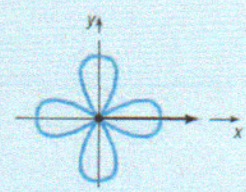
$b - a = 2 - 1 = 1$

$b + a = 2 + 1 = 3$



next page more limaçon graphs

More Equations

Name	Lemniscate figure 8	Rose with three petals	Rose with four petals
Polar equations	$r^2 = a^2\cos(2\theta)$, $a > 0$	$r = a\sin(3\theta)$, $a > 0$	$r = a\sin(2\theta)$, $a > 0$
	$r^2 = a^2\sin(2\theta)$, $a > 0$	$r = a\cos(3\theta)$, $a > 0$	$r = a\cos(2\theta)$, $a > 0$
Typical graph	r^2 		

$a = petal\ length$	number of petals look at coefficient of θ: $\begin{cases} odd = n\ petals \\ even = 2n\ petals \end{cases}$ $a = length\ of\ petal$

what about .... **More on Limaçons!**

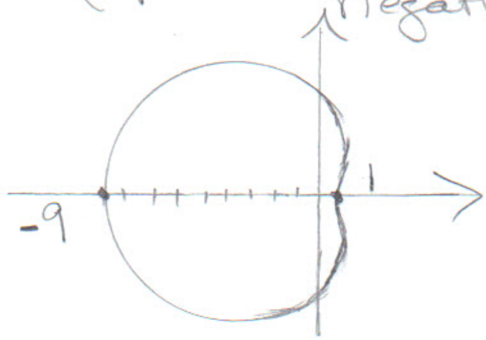
$r = 5 - 4\cos\theta$

$a = 5$ "a" bigger – limaçon – no loop

$b = 4$

(different) cosine along polar axis

Negative ⟹ negative polar axis!



$b + a = 4 + 5 = 9$ negative

$b - a = 4 - 5 = -1$ ⟹ dimple on other side of vertical axis. dimple at 1

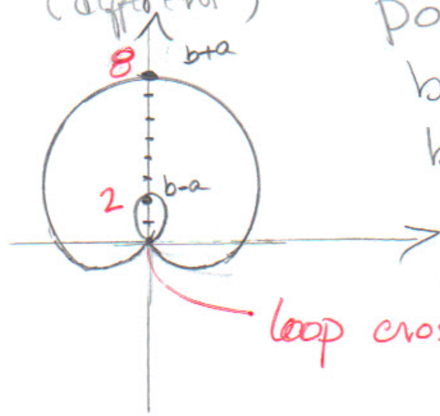
$r = 3 + 5\sin\theta$

$a = 3$ "b" bigger → limaçon w/ loop

$b = 5$

(different) sine along $\frac{\pi}{2}$ axis

positive → positive axis



$b + a = 5 + 3 = 8$

$b - a = 5 - 3 = 2$ – inside of loop

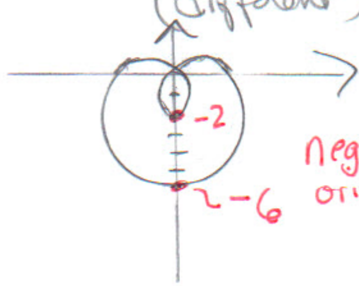
loop crosses here at (0,0)

$r = 2 - 4\sin\theta$

$a = 2$ "b" bigger → limaçon w/ loop

$b = 4$

(different) Negative sine ⟹ limaçon on $-\frac{\pi}{2}$ axis



Negative orientation

$b + a = 4 + 2 = 6$ ⟹ * negative orientation ∴ -6

$b - a = 4 - 2 = 2$ * Negative orientation at -2

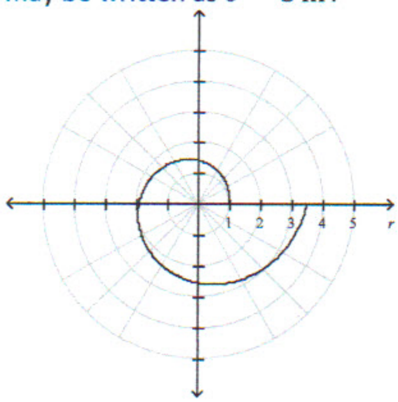
Graphing a Polar Equation (spiral)

It is the locus of points corresponding to the locations over time of a point moving away from a fixed point with a constant speed along a line which rotates with constant angular velocity.

There are several equations that will produce a spiral. The **logarithmic spiral**

$$r = e^{\theta/5}$$

may be written as $\theta = 5\ln r$



Archimedes Spiral is in the form of

$$r = a\theta$$